REGIONALLY PROXIMAL RELATION OF ORDER d IS AN EQUIVALENCE ONE FOR MINIMAL SYSTEMS AND A COMBINATORIAL CONSEQUENCE

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Abstract. By proving the minimality of face transformations acting on the diagonal points and searching the points allowed in the minimal sets, it is shown that the regionally proximal relation of order d, $\mathbf{RP}^{[d]}$, is an equivalence relation for minimal systems. Moreover, the lifting of $\mathbf{RP}^{[d]}$ between two minimal systems is obtained, which implies that the factor induced by $\mathbf{RP}^{[d]}$ is the maximal d-step nilfactor. The above results extend the same conclusions proved by Host, Kra and Maass for minimal distal systems.

A combinatorial consequence is that if S is a dynamically syndetic subset of $\mathbb{Z}$, then for each $d \geq 1$,

$$\{(n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic. In some sense this is the topological correspondence of the result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.

1. Introduction

The background of our study can be seen both in ergodic theory and topological dynamics.

1.1. **Background in ergodic theory.** The connection between ergodic theory and additive combinatorics was built in the 1970's with Furstenberg's beautiful proof of Szemerédi's theorem via ergodic theory [10]. Furstenberg's proof paved the way for obtaining new combinatorial results using ergodic methods, as well as leading to numerous developments within ergodic theory. Roughly speaking, Furstenberg [10] proved Szemerédi's theorem via the following ergodic theorem: Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving transformation on the probability space and let $A \in \mathcal{B}$ with positive measure. Then for every integer $d \geq 1$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-dn}A) > 0.$$

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So it is natural to ask about the convergence of these averages, or more generally about the convergence in $L^2(X,\mu)$ of the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x),$$

where $d \geq 1$ is an integer, $(X,\mathcal{B},\mu,T)$ is a measure preserving system, and $f_1,\dots,f_d \in L^\infty(X,\mu)$. After nearly 30 years' efforts of many researchers, this problem was finally solved in [19, 30].

In their proofs the notion of characteristic factors plays a great role. Let's see why this notion is important. Loosely speaking, the Structure Theorem of [19, 30] states that if one wants to understand the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x),$$

one can replace each function f_i by its conditional expectation on some d-step nilsystem (1-step nilsystem is the Kroneker's one). Thus one can reduce the problem to the study of the same average in a nilsystem, i.e. reducing the average in an arbitrary system to a more tractable question. For example, von Neumann's mean ergodic theorem can be proved by using Kroneker's factor. Note that the multiple ergodic average for commuting transformations was obtained by Tao [26] using finitary ergodic method, see [3, 18] for more traditional ergodic proofs. Unfortunately, in this more general setting, the characteristic factors are not known up till now.

In [19], some useful tools, such as dynamical parallelepipeds, ergodic uniformity seminorms etc., are introduced in the study of dynamical systems. Their further applications were discussed in [18, 20, 21, 22, 23]. Now a natural and important question is what the topological correspondence of characteristic factors is. The history how to obtain the topological counterpart of characteristic factors will be discussed in the next subsection.

1.2. **Background in topological dynamics.** In some sense an equicontinuous system is the simplest system in topological dynamics. In the study of topological dynamics, one of the first problems was to characterize the equicontinuous structure relation $S_{eq}(X)$ of a system (X,T); i.e. to find the smallest closed invariant equivalence relation $R(X)$ on (X,T) such that $(X/R(X),T)$ is equicontinuous. A natural candidate for $R(X)$ is the so-called regionally proximal relation $\mathbf{RP}(X)$ [6]. By the definition $\mathbf{RP}(X)$ is closed, invariant, and reflexive, but not necessarily transitive. The problem was then to find conditions under which $\mathbf{RP}(X)$ is an equivalence relation. It turns out to be a difficult problem. Starting with Veech [27], various authors, including MacMahon [25], Ellis-Keynes [8], came up with various sufficient conditions for $\mathbf{RP}(X)$ to be an equivalence relation. For somewhat different approach, see [2]. Note that in our case, $T: X \to X$ being homeomorphism and (X,T) being minimal, $\mathbf{RP}(X)$ is always an equivalence relation.

In [22, 23] the authors obtained a structure theorem for topological dynamical systems, which can be viewed as an analog of the purely ergodic structure theorem of [10, 19, 30]. Note that previously the counterpart of "characteristic factors" in

topological dynamics was studied by Glasner [14, 15]. In [22, 23], for distal minimal systems a certain generalization of the regionally proximal relation is used to produce the maximal nilfactors.

Here is the notion of the regionally proximal relation of order d defined in [22].

Definition 1.1. Let (X,T) be a system and let $d \geq 1$ be an integer. A pair $(x,y) \in X\times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that $d(x,x') < \delta, d(y,y') < \delta$, and

$$d(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \text{ for any } \epsilon \in \{0,1\}^d,\ \epsilon \neq (0,\dots,0),$$

where $\mathbf{n}\cdot\epsilon = \sum_{i=1}^d \epsilon_i n_i$. The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order d*.

It is easy to see that $\mathbf{RP}^{[d]}(X)$ is a closed and invariant relation for all $d \in \mathbb{N}$. When $d = 1$, $\mathbf{RP}^{[d]}(X)$ is nothing but the classical regionally proximal relation. In [22], for distal minimal systems the authors showed that $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation, and the quotient of X under this relation is its maximal d-step nilfactor. So it remains the question open: is $\mathbf{RP}^{[d]}(X)$ an equivalence relation for any minimal system? The purpose of the current paper is to settle down the question.

1.3. **Main results.** In this article, we show that for all minimal systems $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation and the quotient of X under this relation is its maximal d-step nilfactor.

Note that a subset S of $\mathbb{Z}$ is *dynamically syndetic* if there are a minimal system (X,T), $x \in X$ and an open neighborhood U of x such that $S = \{n \in \mathbb{Z} : T^n x \in U\}$. Equivalently, $S \subset \mathbb{Z}$ is dynamically syndetic if and only if S contains $\{0\}$ and 1_S is a minimal point of $(\{0,1\}^{\mathbb{Z}}, \sigma)$, where σ is the shift map. A subset S of $\mathbb{Z}^d$ is *syndetic* if there exists a finite subset $F \subset \mathbb{Z}^d$ such that $S + F = \mathbb{Z}^d$. A combinatorial consequence of our results is that if S is a dynamically syndetic subset of $\mathbb{Z}$, then for each $d \geq 1$,

$$\{(n_1,\dots,n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic. In some sense this is the topological correspondence of the following result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.

Theorem 1.2. [19, Theorem 1.5] *Let $A \subset \mathbb{Z}$ with $\overline{d}(A) \geq \delta > 0$ and let $d \in \mathbb{N}$, then*

$$\{\mathbf{n} = (n_1, n_2, \dots, n_k) \in \mathbb{Z}^d : \overline{d}\Big(\bigcap_{\epsilon\in\{0,1\}^d} (A + \epsilon\cdot\mathbf{n})\Big) \geq \delta^{2^d}\}$$

is syndetic, where $\overline{d}(B)$ denotes the upper density of $B \subset \mathbb{Z}$.

In [22] the authors showed that the regionally proximal relation of order d is an equivalence relation for minimal distal systems without using the enveloping semigroup theory except one known result that the distal extension between minimal systems is open (which is proved using the theory). In our situation we are forced

to use the theory. The main idea of the proof is the following. First using the structure theory of a minimal system we show that the face transformations acting on the diagonal points are minimal, and then we prove some equivalence condition for two point being regionally proximal of order d. A key lemma here is to switch from a cubic point to a face point. Combining the minimality and the condition we show that the regionally proximal relation of order d is an equivalence relation for minimal systems. Finally we show that $\mathbf{RP}^{[d]}$ can be lifted up from a factor to an extension between two minimal systems, which implies that the factor induced by $\mathbf{RP}^{[d]}$ is the maximal d-step nilfactor.

We remark that many results of the paper can be extended to abelian group actions.

1.4. **Organization of the paper.** In Section 2, we introduce the basic notions used in the paper. Since we will use tools from abstract topological dynamics, we collect basic facts about it in Appendix A. In Section 3, main results of the paper are discussed. The three sections followed are devoted to give proofs of main results. Note that lots of results obtained there have their independent interest. In the final section some applications are given.

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2. Preliminaries

2.1. **Topological dynamical systems.** A *transformation* of a compact metric space X is a homeomorphism of X to itself. A *topological dynamical system*, referred to more succinctly as just a *system*, is a pair (X,T), where X is a compact metric space and $T : X \to X$ is a transformation. We use $d(\cdot,\cdot)$ to denote the metric in X. We also make use of a more general definition of a topological system. That is, instead of just a single transformation T, we will consider a countable abelian group of transformations. We collect basic facts about topological dynamics under general group actions in Appendix A.

A system (X,T) is transitive if there exists some point $x \in X$ whose orbit $\mathcal{O}(x,T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X and we call such a point a transitive point. The system is *minimal* if the orbit of any point is dense in X. This property is equivalent to say that X and the empty set are the only closed invariant sets in X.

2.2. **Cubes and faces.** Let X be a set, let $d \geq 1$ be an integer, and write $[d] = \{1,2,\dots,d\}$. We view $\{0,1\}^d$ in one of two ways, either as a sequence $\epsilon = \epsilon_1 \dots \epsilon_d$ of $0'$s and $1'$s written without commas or parentheses; or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1,\dots,\epsilon_d) \in \{0,1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\mathbf{0} = (0,0,\dots,0) \in \{0,1\}^d$ is the same to $\emptyset \subset [d]$.

If $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0,1\}^d$, we define

$$\mathbf{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i.$$

If we consider ϵ as $\epsilon \subset [d]$, then $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

We denote X^{2^d} by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon : \epsilon \in \{0,1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_\emptyset = x_{\mathbf{0}}$ is the first coordinate of $\mathbf{x}$. As examples, points in $X^{[2]}$ are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}),$$

and points in $X^{[3]}$ are like

$$\begin{aligned} & (x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111}) \\ = \; & (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}, x_3, x_{\{1,3\}}, x_{\{2,3\}}, x_{\{1,2,3\}}). \end{aligned}$$

For $x \in X$, we write $x^{[d]} = (x, x, \ldots, x) \in X^{[d]}$. The diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0,1\}^{d-1})$ and $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0,1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d - 1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\epsilon : \epsilon \neq \emptyset) \in X_*^{[d]}$.

Identifying $\{0,1\}^d$ with the set of vertices of the Euclidean unit cube, a Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $x \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$.

2.3. **Dynamical parallelepipeds.**

Definition 2.1. Let (X, T) be a topological dynamical system and let $d \geq 1$ be an integer. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\mathbf{n} \cdot \epsilon} x = T^{n_1 \epsilon_1 + \ldots + n_d \epsilon_d} x : \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d),$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. When there is no ambiguity, we write $\mathbf{Q}^{[d]}$ instead of $\mathbf{Q}^{[d]}(X)$. An element of $\mathbf{Q}^{[d]}(X)$ is called a (dynamical) *parallelepiped of dimension d*.

It is important to note that $\mathbf{Q}^{[d]}$ is invariant under the Euclidean permutations of $X^{[d]}$.

As examples, $\mathbf{Q}^{[2]}$ is the closure in $X^{[2]} = X^4$ of the set

$$\{(x, T^m x, T^n x, T^{n+m} x) : x \in X, m, n \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}$ is the closure in $X^{[3]} = X^8$ of the set

$$\{(x, T^m x, T^n x, T^{m+n} x, T^p x, T^{m+p} x, T^{n+p} x, T^{m+n+p} x) : x \in X, m, n, p \in \mathbb{Z}\}.$$

Definition 2.2. Let $\phi : X \to Y$ and $d \in \mathbb{N}$. Define $\phi^{[d]} : X^{[d]} \to Y^{[d]}$ by $(\phi^{[d]}\mathbf{x})_\epsilon = \phi x_\epsilon$ for every $\mathbf{x} \in X^{[d]}$ and every $\epsilon \subset [d]$.

Let (X,T) be a system and $d \geq 1$ be an integer. The *diagonal transformation* of $X^{[d]}$ is the map $T^{[d]}$.

Definition 2.3. *Face transformations* are defined inductively as follows: Let $T^{[0]} = T$, $T_1^{[1]} = \mathrm{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is defined already, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]},\ j \in \{1,2,\dots,d-1\},$$
$$T_d^{[d]} = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $j \in [d]$, the face transformation $T_j^{[d]} : X^{[d]} \to X^{[d]}$ can be defined by, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \subset [d]$,

$$T_j^{[d]}\mathbf{x} = \begin{cases} (T_j^{[d]}\mathbf{x})_\epsilon = Tx_\epsilon, & j \in \epsilon; \\ (T_j^{[d]}\mathbf{x})_\epsilon = x_\epsilon, & j \notin \epsilon. \end{cases}$$

The *face group* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For $\mathcal{G}^{[d]}$ and $\mathcal{F}^{[d]}$, we use similar notations to that used for $X^{[d]}$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0,1\}^d)$. In particular, $\mathcal{F}^{[d]} = \{S \in \mathcal{G}^{[d]} : S_\emptyset = \mathrm{id}\}$.

For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$.

It is easy to verify that $\mathbf{Q}^{[d]}$ is the closure in $X^{[d]}$ of

$$\{Sx^{[d]} : S \in \mathcal{F}^{[d]}, x \in X\}.$$

If x is a transitive point of X, then $\mathbf{Q}^{[d]}$ is the closed orbit of $x^{[d]}$ under the group $\mathcal{G}^{[d]}$.

2.4. **Nilmanifolds and nilsystems.** Let G be a group. For $g, h \in G$, we write $[g,h] = ghg^{-1}h^{-1}$ for the commutator of g and h and we write $[A,B]$ for the subgroup spanned by $\{[a,b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $k \geq 1$ be an integer. We say that G is *k-step nilpotent* if G_{k+1} is the trivial subgroup.

Let G be a k-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a *k-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g,x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$ of X. Then (X,T,μ) is called a *basic k-step nilsystem*. When the measure is not needed for results, we omit and write that (X,T) is a basic k-step nilsystem.

We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse

limits). If $(X_i,T_i)_{i\in\mathbb{N}}$ are systems with $diam(X_i)\le 1$ and $\phi_i: X_{i+1}\to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i\in\mathbb{N}\}$, which is denoted by $\overleftarrow{\lim}\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $d(x,y)=\sum_{i\in\mathbb{N}} 1/2^i d_i(x_i,y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

Theorem 2.4 (Host-Kra-Maass)**.** [22, Theorem 1.2] *Assume that (X,T) is a transitive topological dynamical system and let $d\ge 2$ be an integer. The following properties are equivalent:*

(1) *If $x,y\in \mathbf{Q}^{[d]}(X)$ have 2^d-1 coordinates in common, then $x=y$.*
(2) *If $x,y\in X$ are such that $(x,y,\dots,y)\in \mathbf{Q}^{[d]}(X)$, then $x=y$.*
(3) *X is an inverse limit of basic $(d-1)$-step minimal nilsystems.*

A transitive system satisfying either of the equivalent properties above is called a $(d-1)$-step nilsystem *or a* system of order $(d-1)$.

2.5. **Definition of the regionally proximal relations.**

Definition 2.5. Let (X,T) be a system and let $d\ge 1$ be an integer. A pair $(x,y)\in X\times X$ is said to be *regionally proximal of order* d if for any $\delta>0$, there exist $x',y'\in X$ and a vector $\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d$ such that $d(x,x')<\delta, d(y,y')<\delta$, and
$$d(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y')<\delta \text{ for any nonempty } \epsilon\subset[d].$$
(In other words, there exists $S\in\mathcal{F}^{[d]}$ such that $d(S_\epsilon x', S_\epsilon y')<\delta$ for every $\epsilon\neq\emptyset$.) The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}$ (or by $\mathbf{RP}^{[d]}(X)$ in case of ambiguity), which is called *the regionally proximal relation of order d*.

It is easy to see that $\mathbf{RP}^{[d]}$ is a closed and invariant relation for all $d\in\mathbb{N}$. Note that
$$\ldots\subseteq \mathbf{RP}^{[d+1]}\subseteq \mathbf{RP}^{[d]}\subseteq\ldots\mathbf{RP}^{[2]}\subseteq \mathbf{RP}^{[1]}=\mathbf{RP}(X).$$
By the definition it is easy to verify the following equivalent condition for $\mathbf{RP}^{[d]}$, see [22].

Lemma 2.6. *Let (X,T) be a minimal system and let $d\ge 1$ be an integer. Let $x,y\in X$. Then $(x,y)\in\mathbf{RP}^{[d]}$ if and only if there is some $\mathbf{a}_*\in X_*^{[d]}$ such that $(x,\mathbf{a}_*,y,\mathbf{a}_*)\in\mathbf{Q}^{[d+1]}$.*

Remark 2.7. When $d=1$, $\mathbf{RP}^{[1]}$ is the classical regionally proximal relation. If (X,T) is minimal, it is easy to verify directly the following useful fact:
$$(x,y)\in\mathbf{RP}=\mathbf{RP}^{[1]}\Leftrightarrow (x,x,y,x)\in\mathbf{Q}^{[2]}\Leftrightarrow (x,y,y,y)\in\mathbf{Q}^{[2]}.$$

3. Main results

In this section we will state the main results of the paper.

3.1. **$\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$.** To show $\mathbf{RP}^{[d]}$ is an equivalence relation we are forced to investigate the $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and the equivalent conditions for $\mathbf{RP}^{[d]}$. Those are done in Theorem 3.1 and Theorem 3.2 respectively.

First recall that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system, which is mentioned in [22]. But we need to know $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$. Let (X,T) be a system and $x \in X$. Recall that $\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$ for $\mathbf{x} \in X^{[d]}$. Set

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

Theorem 3.1. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then*

(1) *$(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*
(2) *$(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$ for all $x \in X$.*

3.2. **$\mathbf{RP}^{[d]}$ is an equivalence relation.** With the help of Theorem 3.1, we can prove that $\mathbf{RP}^{[d]}$ is an equivalence relation. First we have the following equivalent conditions for $\mathbf{RP}^{[d]}$.

Theorem 3.2. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *$(x,y) \in \mathbf{RP}^{[d]}$;*
(2) *$(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$;*
(3) *$(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.*

Proof. (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1) follows from Lemma 2.6. Hence it suffices to show (1) $\Rightarrow$ (3).

Let $(x,y) \in \mathbf{RP}^{[d]}$. Then by Lemma 2.6 there is some $\mathbf{a}_* \in X_*^{[d]}$ such that $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$. Observe that $(y, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$. By Theorem 3.1-(2), there is a sequence $\{F_k\} \subset \mathcal{F}^{[d]}$ such that $F_k(y, \mathbf{a}_*) \to y^{[d]}, k \to \infty$. Hence

$$F_k \times F_k(x, \mathbf{a}_*, y, \mathbf{a}_*) \to (x, y_*^{[d]}, y, y_*^{[d]}) = (x, y_*^{[d+1]}), \ k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$, we have that $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$.

By Theorem 3.1-(1), $y^{[d+1]}$ is $\mathcal{F}^{[d+1]}$-minimal. It follows that $(x, y_*^{[d+1]})$ is also $\mathcal{F}^{[d+1]}$-minimal. Now $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}[x]$ and by Theorem 3.1-(2), $(\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]}), \mathcal{F}^{[d+1]})$ is the unique $\mathcal{F}^{[d+1]}$-minimal subset in $\mathbf{Q}^{[d+1]}[x]$. Hence we have that $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$, and the proof is completed. $\square$

By Theorem 3.2, we have the following theorem immediately.

Theorem 3.3. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]}(X)$ is an equivalence relation.*

Proof. Let $(x,y),(y,z) \in \mathbf{RP}^{[d]}(X)$. By Theorem 3.2, we have

$$(y,x,x,\dots,x), (y,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}).$$

By Theorem 3.1 $(\overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal, it follows that $(y,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(y,x,x,\dots,x)$. Thus $(x,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. By Theorem 3.2, $(x,z) \in \mathbf{RP}^{[d]}(X)$. $\square$

Remark 3.4. By Theorem 3.2 we know that in the definition of regionally proximal relation of d, x' can be replaced by x. More precisely, $(x, y) \in \mathbf{RP}^{[d]}$ if and only if for any $\delta > 0$ there exist $y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that for any nonempty $\epsilon \subset [d]$

$$d(y, y') < \delta \text{ and } d(T^{\mathbf{n}\cdot\epsilon}x, T^{\mathbf{n}\cdot\epsilon}y') < \delta.$$

3.3. **$\mathbf{RP}^{[d]}$ and nilfactors.** $S \subset \mathbb{Z}$ is *thick* if it contains arbitrarily long runs of positive integers, i.e. there is a subsequence $\{n_i\}$ of $\mathbb{Z}$ such that $S \supset \bigcup_{i=1}^{\infty}\{n_i, n_i + 1, \ldots, n_i + i\}$.

Let $\{b_i\}_{i\in I}$ be a finite or infinite sequence in $\mathbb{Z}$. One defines

$$FS(\{b_i\}_{i\in I}) = \Big\{ \sum_{i\in\alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \Big\}$$

Note when $I = [d]$,

$$FS(\{b_i\}_{i=1}^d) = \Big\{ \sum_{i\in I} b_i\epsilon_i : \epsilon = (\epsilon_i) \in \{0,1\}^d \setminus \{\emptyset\} \Big\}.$$

F is an *IP set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, where $p_i \in \mathbb{Z}$.

Lemma 3.5. *Let (X, T) be a system. Then for every $d \in \mathbb{N}$, the proximal relation*

$$\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X).$$

Proof. Let $(x, y) \in \mathbf{P}(X)$ and $\delta > 0$. Set

$$N_\delta(x, y) = \{n \in \mathbb{Z} : d(T^n x, T^n y) < \delta\}.$$

It is easy to check $N_\delta(x, y)$ is thick and hence an IP set. From this it follows that $\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X)$. More precisely, set $FS(\{p_i\}_{i=1}^{\infty}) \subseteq N_\delta(x, y)$, then for any $d \in \mathbb{N}$,

$$d(T^{p_1\epsilon_1+\ldots+p_d\epsilon_d}x, T^{p_1\epsilon_1+\ldots+p_d\epsilon_d}y) < \delta,\ \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d, \epsilon \neq (0, \ldots, 0).$$

That is, $(x, y) \in \mathbf{RP}^{[d]}$ for all $d \in \mathbb{N}$. □

The following corollary was observed in [23] for $d = 2$.

Corollary 3.6. *If (X, T) is a weakly mixing system, then for every $d \in \mathbb{N}$,*

$$\mathbf{RP}^{[d]} = X \times X.$$

Proof. Since a system (X, T) is weakly mixing if and only if $\overline{\mathbf{P}(X)} = X \times X$ (see [1]), so the result follows from Lemma 3.5. □

We remark that more interesting properties for weakly mixing systems will be shown in Theorem 3.11 in the sequel.

Proposition 3.7. *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]} = \Delta$ if and only if X is a system of order d.*

Proof. It follows from Theorem 3.2 and Theorem 2.4 directly. □

3.4. **Maximal nilfactors.** Note that the lifting property of $\mathbf{RP}^{[d]}$ between two minimal systems is obtained in the paper. This result is new even for minimal distal systems.

Theorem 3.8. *Let $\pi : (X,T) \to (Y,T)$ be a factor map and $d \in \mathbb{N}$. Then*
(1) $\pi \times \pi(\mathbf{RP}^{[d]}(X)) \subseteq \mathbf{RP}^{[d]}(Y)$;
(2) *if (X,T) is minimal, then* $\pi \times \pi(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.

Proof. (1) It follows from the definition.
(2) It will be proved in Section 6. □

Theorem 3.9. *Let $\pi : (X,T) \to (Y,T)$ be a factor map of minimal systems and $d \in \mathbb{N}$. Then the following conditions are equivalent:*
(1) *(Y,T) is a system of order d;*
(2) $\mathbf{RP}^{[d]}(X) \subset R_\pi$.

Especially the quotient of X under $\mathbf{RP}^{[d]}(X)$ is the maximal d-step nilfactor of X, i.e. any d-step nilfactor of X is the factor of $X/\mathbf{RP}^{[d]}(X)$.

Proof. Assume that (Y,T) is a system of order d. Then we have $\mathbf{RP}^{[d]}(Y) = \Delta_Y$ by Proposition 3.7. Hence by Theorem 3.8-(1),

$$\mathbf{RP}^{[d]}(X) \subset (\pi \times \pi)^{-1}(\Delta_Y) = R_\pi.$$

Conversely, assume that $\mathbf{RP}^{[d]}(X) \subset R_\pi$. If (Y,T) is not a system of order d, then by Proposition 3.7, $\mathbf{RP}^{[d]}(Y) \neq \Delta_Y$. Let $(y_1,y_2) \in \mathbf{RP}^{[d]} \setminus \Delta_Y$. Now by Theorem 3.8, there are $x_1,x_2 \in X$ such that $(x_1,x_2) \in \mathbf{RP}^{[d]}(X)$ with $(\pi \times \pi)(x_1,x_2) = (y_1,y_2)$. Since $\pi(x_1) = y_1 \neq y_2 = \pi(x_2)$, $(x_1,x_2) \notin R_\pi$. This means that $\mathbf{RP}^{[d]}(X) \not\subset R_\pi$, a contradiction! The proof is completed. □

Remark 3.10. In [22, Proposition 4.5] it is showed that this proposition holds for minimal distal systems.

3.5. **Weakly mixing systems.** In this subsection we completely determine $\mathbf{Q}^{[d]}$ and $\overline{\mathcal{F}^{[d]}}(x^{[d]})$ for minimal weakly mixing systems.

Theorem 3.11. *Let (X,T) be a minimal weakly mixing system and $d \geq 1$. Then*
(1) *$(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$;*
(2) *For all $x \in X$, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal and*

$$\overline{\mathcal{F}^{[d]}}(x^{[d]}) = \{x\} \times X_*^{[d]} = \{x\} \times X^{2^d-1}.$$

Proof. The fact that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$ is followed from (2) easily. Hence it suffices to show (2).

We will show for any point of $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$, we have

$$\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \{x\} \times X_*^{[d]},$$

which obviously implies (2). First note that it is trivial for $d = 1$. Now we assume that it holds for $d-1$, $d \geq 2$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbf{Q}^{[d]}$. Since (X,T) is weakly mixing, $(X^{[d-1]}, T^{[d-1]})$ is transitive (see [9]). Let $\mathbf{a} \in X^{[d-1]}$ be a transitive point. By the induction for $d-1$, $\mathbf{Q}^{[d-1]} =$

$X^{[d-1]}$ is $\mathcal{G}^{[d]}$-minimal. Hence $\mathbf{a} \in \overline{\mathcal{O}(\mathbf{x}'', \mathcal{G}^{[d-1]})}$ and there is some sequence $F_k \in \mathcal{F}^{[d]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k \mathbf{x} = F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}),\ k \to \infty.$$

Especially $(\mathbf{w}, \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Note that

$$(T_d^{[d]})^n(\mathbf{w}, \mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x})$$

And so

$$\{\mathbf{w}\} \times X^{[d-1]} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.1}$$

By the induction assumption for $d-1$, $\mathbf{w}$ is minimal for $\mathcal{F}^{[d-1]}$ action and

$$\overline{\mathcal{F}^{[d-1]}}(\mathbf{w}) = \overline{\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]})} = \{x\} \times X_*^{[d-1]}. \tag{3.2}$$

By acting the elements of $\mathcal{F}^{[d]}$ on (3.1), we have

$$\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]}) \times X^{[d-1]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\{x\} \times X_*^{[d-1]} \times X^{[d-1]} = \{x\} \times X_*^{[d]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

This completes the proof. □

4. $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$

In this section we discuss $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and prove Theorem 3.1-(1). First we will discuss proximal extensions, distal extensions and weakly mixing extension one by one. They exhibit different properties and satisfy our requests by different reasons. After that, the proof of Theorem 3.1-(1) will be given. The proof of Theorem 3.1-(2) will be given in next section. For notions which are not mentioned before see Appendix A.

4.1. **Idea of the proof of Theorem 3.1-(1).** Before going on let us say something about the idea in the proof of Theorem 3.1-(1). By the structure theorem A.6, for a minimal system (X, T), we have the following diagram.

$$\begin{tikzcd} X_\infty \arrow[r, "\pi"] \arrow[d, "\phi"] & X \\ Y_\infty & \end{tikzcd}$$

In this diagram Y_∞ is a strictly PI system, ϕ is weakly mixing and RIC, and π is proximal.

So if we want to show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$, it is sufficient to show it holds for X_∞. By the definition of X_∞ and Y_∞, it is sufficient to consider the following cases: (1) proximal extensions; (2) distal or equicontinuous extensions; (3) RIC weakly mixing extensions and (4) the inverse limit. Since the inverse limit is easy to handle, we need only focus on the three different extensions.

4.2. **Properties about three kinds of extensions.** In this subsection we collect some properties about proximal, distal and weakly mixing extensions, which will be used frequently in the sequel. As in Appendix A, $(X,\mathcal{T})$ is a system under the action of a topological group $\mathcal{T}$, and $E(X,\mathcal{T})$ is its enveloping semigroup.

The following two lemmas are folk results, for completeness we include proofs.

Lemma 4.1. *Let $\pi:(X,\mathcal{T})\to(Y,\mathcal{T})$ be a proximal extension of minimal systems. Let $x\in X, y=\pi(x)$ and let $x_1,x_2,\ldots,x_n\in\pi^{-1}(y)$. Then there is some $p\in E(X,\mathcal{T})$ such that*

$$px_1=px_2=\ldots=px_n=x.$$

Especially, when $x=x_1$, we have that $(x_1,x_2,\ldots,x_n)$ is proximal to $(x,x,\ldots,x)$ in $(X^n,\mathcal{T})$.

Proof. Since $(x_1,x_2)\in R_\pi\subset\mathbf{P}(X,\mathcal{T})$, by Proposition A.3 there is some $p\in E(X,\mathcal{T})$ such that $px_1=px_2$.

Now assume that for $2\le j\le n-1$, there is some $p_1\in E(X,\mathcal{T})$ such that $p_1x_1=p_1x_2=\ldots=p_1x_j$. Since R_π is closed and invariant and $(x_j,x_{j+1})\in R_\pi$, $(p_1x_j,p_1x_{j+1})\in R_\pi\subset\mathbf{P}(X,\mathcal{T})$. So by Proposition A.3 there is $p_2\in E(X,\mathcal{T})$ such that $p_2(p_1x_j)=p_2(p_1x_{j+1})$. Let $p=p_2p_1$, then we have

$$px_1=px_2=\ldots=px_j=px_{j+1}.$$

Inductively, there is some $p\in E(X,\mathcal{T})$ such that

$$px_1=px_2=\ldots=px_n.$$

Since $(X,\mathcal{T})$ is minimal, we can assume that they are equal to x.

If $x_1=x$, then $px_1=px_2=\ldots=px_n=x=x_1$ and hence

$$p(x_1,x_2,\ldots,x_n)=(x,x,\ldots,x)=p(x,x,\ldots,x).$$

That is, $(x_1,x_2,\ldots,x_n)$ is proximal to $(x,x,\ldots,x)$ in $(X^n,\mathcal{T})$. □

Lemma 4.2. *Let $\pi:(X,\mathcal{T})\to(Y,\mathcal{T})$ be a distal extension of systems. Then for any $x\in X$, if $\pi(x)$ is minimal in $(Y,\mathcal{T})$, then x is minimal in $(X,\mathcal{T})$. Especially, if $(Y,\mathcal{T})$ is semi-simple (i.e. every point is minimal), then so is $(X,\mathcal{T})$.*

Proof. Let $x\in X$ and $y=\pi(x)$. Since y is a minimal point, by Proposition A.2 there is some minimal idempotent $u\in E(X,\mathcal{T})$ such that $uy=y$. Then $\pi(ux)=u\pi(x)=uy=y$. Hence $ux,x\in\pi^{-1}(y)$. Since $(ux,x)\in\mathbf{P}(X,\mathcal{T})$ (Proposition A.3) and π is distal, we have $ux=x$. That is, x is a minimal point of X by Proposition A.2. □

Now we discuss weakly mixing extensions. We need Theorem 4.3, which is a generalization of [1, Chapter 14, Theorem 28]. Note that in [17, Theorem 2.7 and Corollary 2.9] Glasner showed that R_π^n is transitive. So Theorem 4.3 is a slightly strengthen of the results in [17]. Since its proof needs some techniques in the enveloping semigroup theory, we leave it to the appendix.

Theorem 4.3. *Let $\pi:(X,\mathcal{T})\to(Y,\mathcal{T})$ be a RIC weakly mixing extension of minimal systems, then for all $n\ge1$ and $y\in Y$, there exists a transitive point $(x_1,x_2,\ldots,x_n)$ of R_π^n with $x_1,x_2,\ldots,x_n\in\pi^{-1}(y)$.*

Note that each RIC extension is open, and $\pi : X \to Y$ is open if and only if $Y \to 2^X, y \mapsto \pi^{-1}(y)$ is continuous, see for instance [29]. Using Theorem 4.3 we have the following lemma, which will be used in the sequel.

Lemma 4.4. *Let $\pi : (X,T) \to (Y,T)$ be a RIC weakly mixing extension of minimal systems of (X,T) and (Y,T). Then for each $y \in Y$ and $d \geq 1$, we have*

(1) $(\pi^{-1}(y))^{[d]} = (\pi^{-1}(y))^{2^d} \subset \mathbf{Q}^{[d]}(X)$,

(2) *for all* $\mathbf{x} \in X^{[d]}$ *with* $x_\emptyset = x$ *and* $\pi^{[d]}(\mathbf{x}) = y^{[d]}$

$$\{x\} \times \left(\pi^{-1}(y)\right)_*^{[d]} = \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Proof. The idea of proof is similar to Theorem 3.11. When $d=1$, for any $(x,x') \in X^{[1]} = X \times X$, $\overline{\mathcal{F}^{[1]}}(x,x') = \overline{\mathcal{O}}\left((x,x'), \mathrm{id} \times T\right) = \{x\} \times X$ and $\mathbf{Q}^{[1]}(X) = X \times X$. Hence the results hold obviously. Now we show the case for $d=2$. Let $\mathbf{x} = (x_1,x_2,x_3,x_4) \in X^{[2]}$ with $\pi^{[2]}(x_1,x_2,x_3,x_4) = y^{[2]}$. By Theorem 4.3, there is a transitive point (a,b) of $(R_\pi, T\times T)$ with $\pi(a) = \pi(b) = y$. Since (X,T) is minimal, there is some sequence $\{n_i\} \subset \mathbb{Z}$ such that $T^{n_i}x_3 \to a, i \to \infty$. Without loss of generality, assume that $T^{n_i}x_4 \to x_4', i \to \infty$ for some $x_4' \in X$. Since $\pi(a) = y$, $\pi(x_4') = y$ too. So

$$(\mathrm{id} \times \mathrm{id} \times T \times T)^{n_i}(x_1,x_2,x_3,x_4) \to (x_1,x_2,a,x_4'),\ i \to \infty. \tag{4.1}$$

Since (X,T) is minimal, there is some sequence $\{m_i\} \subset \mathbb{Z}$ such that $T^{m_i}x_4' \to b, i \to \infty$. Without loss of generality, assume that $T^{m_i}x_2 \to x_2', i \to \infty$ for some $x_2' \in X$. Since $\pi(b) = y$, $\pi(x_2') = y$ too. So

$$(\mathrm{id} \times T \times \mathrm{id} \times T)^{m_i}(x_1,x_2,a,x_4') \to (x_1,x_2',a,b),\ i \to \infty. \tag{4.2}$$

Hence by (4.1) and (4.2),

$$(x_1,x_2',a,b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.3}$$

Thus for all $n \in \mathbb{Z}$,

$$(x_1,x_2',T^na,T^nb) = (\mathrm{id} \times \mathrm{id} \times T \times T)^n(x_1,x_2',a,b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since (a,b) is a transitive point of $(R_\pi, T\times T)$, it follows that

$$\{x_1\} \times \{x_2'\} \times \pi^{-1}(y) \times \pi^{-1}(y) \subset \{x_1\} \times \{x_2'\} \times R_\pi \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.4}$$

Now we show that

$$\{x_1\} \times \pi^{-1}(y) \times \pi^{-1}(y) \times \pi^{-1}(y) = \{x_1\} \times (\pi^{-1}(y))^3 \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.5}$$

For any $z \in \pi^{-1}(y)$, there is a sequence $k_i \subset \mathbb{Z}$ such that $T^{k_i}x_2' \to z, i \to \infty$. Thus $T^{k_i}y = T^{k_i}\pi(x_2') = \pi(T^{k_i}x_2') \to \pi(z) = y, i \to \infty$. Since π is open, we have $T^{k_i}\pi^{-1}(y) = \pi^{-1}(T^{k_i}y) \to \pi^{-1}(y), i \to \infty$ in the Hausdorff metric. Thus

$$\{x_1\} \times \{z\} \times \pi^{-1}(y)^2 \subset \overline{\cup_{i=1}^\infty (\mathrm{id} \times T \times \mathrm{id} \times T)^{k_i}(\{x_1\} \times \{x_2'\} \times \pi^{-1}(y)^2)} \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since z is arbitrary, we have (4.5). Similarly, we have $\left(\pi^{-1}(y)\right)^4 \subset \mathbf{Q}^{[2]}(X)$ and we are done for $d=2$.

Now assume we have (1) and (2) for $d-1$ already, and show the case for d. Let $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$ and $\pi^{[d]}(\mathbf{x}) = y^{[d]}$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$. Since π is weakly mixing, $(R_\pi^{2^{d-1}}, T^{[d-1]})$ is transitive. By Theorem 4.3 there is $\mathbf{a} \in R_\pi^{2^{d-1}}$ which is a transitive point of $(R_\pi^{2^{d-1}}, T^{[d-1]})$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$. Without loss of generality, we may assume that $a_\emptyset = x''_\emptyset$ (i.e. the first coordinate of $\mathbf{a}$ is equal to that of $\mathbf{x}''$), otherwise we may use the face transformation $\mathrm{id}^{[d-1]} \times T^{[d-1]}$ to find some point in $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$ satisfying this property.

By the induction assumption for $d-1$,

$$\mathbf{a} \in \{x''_\emptyset\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{x}'').$$

Hence there is some sequence $F_k \in \mathcal{F}^{[d-1]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k \times F_k(\mathbf{x}) = F_k \times F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}),\ k \to \infty.$$

Especially $(\mathbf{w}, \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Since $\pi^{[d]}(\mathbf{x}) = y^{[d]}$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$, it is easy to verify that $\pi^{[d-1]}(\mathbf{w}) = y^{[d-1]}$ and $w_\emptyset = x$. Note that

$$(T_d^{[d]})^n(\mathbf{w}, \mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

And so

$$\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} \subset \{\mathbf{w}\} \times R_\pi^{2^{d-1}} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{4.6}$$

By the induction assumption for $d-1$, for $\mathbf{w}$ we have

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{w}). \tag{4.7}$$

Hence for all $\mathbf{z} \in \{x\} \times (\pi^{-1}(y))^{2^{d-1}-1}$, there is some sequence $\{H_k\} \subset \mathcal{F}^{[d-1]}$ such that $H_k \mathbf{w} \to \mathbf{z}, k \to \infty$. Since π is open, similar to the proof of (4.5), we have that $H_k\,(\pi^{-1}(y))^{2^{d-1}} \to (\pi^{-1}(y))^{2^{d-1}}, k \to \infty$. Hence

$$H_k \times H_k\ \left(\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}\right) \to \{\mathbf{z}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}, k \to \infty.$$

Since $H_k \times H_k \in \mathcal{F}^{[d]}$ and $\mathbf{z} \in \{x\} \times (\pi^{-1}(y))^{2^{d-1}-1}$ is arbitrary, it follows from (4.6) that

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} = \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Now by this fact it is easy to get $(\pi^{-1}(y))^{[d]} = (\pi^{-1}(y))^{2^d} \subset \mathbf{Q}^{[d]}(X)$. So (1) and (2) hold for the case d. This completes the proof. $\square$

In fact with a small modification of the above proof one can show that $R_\pi^{2^d} \subset \mathbf{Q}^{[d]}(X)$. We do not know if $\{x\} \times R_\pi^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x})$.

4.3. **Proof of Theorem 3.1-(1).** A subset $S \subseteq \mathbb{Z}$ is a *central set* if there exists a system (X, T), a point $x \in X$ and a minimal point $y \in X$ proximal to x, and a neighborhood U_y of y such that $N(x, U_y) \subset S$. It is known that any central set is an IP-set [11, Proposition 8.10.].

Proposition 4.5. *Let $\pi : (X, T) \to (Y, T)$ be a proximal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Especially there is some $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k\mathbf{y} \to y^{[d]},\ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k\mathbf{x} \to \mathbf{z},\ k \to \infty. \tag{4.8}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k\mathbf{x}) = \lim_k F_k\mathbf{y} = y^{[d]}$. That is,

$$z_\epsilon \in \pi^{-1}(y),\ \forall \epsilon \in \{0,1\}^d.$$

Since π is proximal, by Lemma 4.1 there is some $p \in E(X,T)$ such that

$$pz_\epsilon = px = x, \quad \forall \epsilon \in \{0,1\}^d.$$

That is, $p\mathbf{z} = x^{[d]} = px^{[d]}$, i.e. $\mathbf{z}$ is proximal to $x^{[d]}$ under the action of $T^{[d]}$. Since $x^{[d]}$ is $T^{[d]}$-minimal, for any neighborhood $\mathbf{U}$ of $x^{[d]}$,

$$N_{T^{[d]}}(\mathbf{z}, \mathbf{U}) = \{n \in \mathbb{Z} : (T^{[d]})^n\mathbf{z} \in \mathbf{U}\}$$

is a central set and hence contains some IP set $FS(\{p_i\}_{i=1}^\infty)$. Particularly,

$$FS(\{p_i\}_{i=1}^d) \subseteq N_{T^{[d]}}(\mathbf{z}, \mathbf{U}).$$

This means for all $\epsilon \in \{0,1\}^d$,

$$(T^{[d]})^{\mathbf{p}\cdot\epsilon}\mathbf{z} \in \mathbf{U},$$

where $\mathbf{p} = (p_1, p_2, \dots, p_d) \in \mathbb{Z}^d$. Especially,

$$(T^{\mathbf{p}\cdot\epsilon}z_\epsilon)_{\epsilon\in\{0,1\}^d} \in \mathbf{U}$$

In other words, we have

$$(T_1^{[d]})^{p_1}(T_2^{[d]})^{p_2}\dots(T_d^{[d]})^{p_d}\mathbf{z} \in \mathbf{U}.$$

Since $\mathbf{U}$ is arbitrary, we have that $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{z})$. Combining with (4.8), we have

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Thus $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. This completes the proof. $\square$

Proposition 4.6. *Let $\pi : (X,T) \to (Y,T)$ be a distal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It follows from Lemma 4.2, since $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a distal extension. $\square$

Proposition 4.7. *Let $\pi : (X,T) \to (Y,T)$ be a RIC weakly mixing extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Especially there is some $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k\mathbf{y} \to y^{[d]},\ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k\mathbf{x} \to \mathbf{z},\ k \to \infty. \tag{4.9}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k\mathbf{x}) = \lim_k F_k\mathbf{y} = y^{[d]}$. By Lemma 4.4

$$x^{[d]} \in \{x\} \times \big(\pi^{-1}(y)\big)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{z}).$$

Together with (4.9), we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. This completes the proof. $\square$

Proof of Theorem 3.1-(1): By the structure theorem A.6, we have the following diagram, where Y_∞ is a strictly PI-system, ϕ is RIC weakly mixing extension and π is proximal.

$$\begin{array}{ccc} X_\infty & \xrightarrow{\pi} & X \\ \big\downarrow{\scriptstyle\phi} & & \\ Y_\infty & & \end{array}$$

Since the inverse limit of minimal systems is minimal, it follows from Propositions 4.5, 4.6 that the result holds for Y_∞. By Proposition 4.7 it also holds for X_∞. Since the factor of a minimal system is always minimal, it is easy to see that we have the theorem for X. $\square$

4.4. Minimality of $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$.

We will need the following theorem mentioned in [22], where no proof is included. We give a proof (due to Glasner-Ellis) here for completeness. Note one can also prove this result using the method in the previous subsection.

Proposition 4.8. *Let (X, T) be a minimal system and let $d \geq 1$ be an integer. Let A be a $T^{[d]}$-minimal subset of $X^{[d]}$ and set $N = \overline{\mathcal{O}(A, \mathcal{F}^{[d]})} = \mathrm{cl}\big(\bigcup\{SA : S \in \mathcal{F}^{[d]}\}\big)$. Then $(N, \mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in N.*

Proof. The proof is similar to the one in [16]. Let $E = E(N, \mathcal{G}^{[d]})$ be the enveloping semigroup of $(N, \mathcal{G}^{[d]})$. Let $\pi_\epsilon : N \to X$ be the projection of N on the ϵ-th component, $\epsilon \in \{0,1\}^d$. We consider the action of the group $\mathcal{G}^{[d]}$ on the ϵ-th component via the representation $T^{[d]} \mapsto T$ and

$$T_j^{[d]} \mapsto \begin{cases} T, & j \in \epsilon; \\ \mathrm{id}, & j \notin \epsilon. \end{cases}$$

With respect to this action of $\mathcal{G}^{[d]}$ on X the map π_ϵ is a factor map $\pi_\epsilon : (N, \mathcal{G}^{[d]}) \to (X, \mathcal{G}^{[d]})$. Let $\pi_\epsilon^* : E(N, \mathcal{G}^{[d]}) \to E(X, \mathcal{G}^{[d]})$ be the corresponding homomorphism of

enveloping semigroups. Notice that for this action of $\mathcal{G}^{[d]}$ on X clearly $E(X,\mathcal{G}^{[d]})=E(X,T)$ as subsets of X^X.

Let now $u\in E(N,T^{[d]})$ be any minimal idempotent in the enveloping semigroup of $(N,T^{[d]})$. Choose v a minimal idempotent in the closed left ideal $E(N,\mathcal{G}^{[d]})u$. Then $vu=v$, i.e. $v<_L u$. Set for each $\epsilon\in\{0,1\}^d$, $u_\epsilon=\pi_\epsilon^* u$ and $v_\epsilon=\pi^*_\epsilon v$. We want to show that also $uv=u$, i.e. $u<_L v$. Note that as an element of $E(N,\mathcal{G}^{[d]})$ is determined by its projections, it suffices to show that for each $\epsilon\in\{0,1\}^d$, $u_\epsilon v_\epsilon=u_\epsilon$.

Since for each $\epsilon\in\{0,1\}^d$ the map π^*_ϵ is a semigroup homomorphism, we have $v_\epsilon u_\epsilon=v_\epsilon$ as $vu=v$. In particular we deduce that v_ϵ is an element of the minimal left ideal of $E(X,T)$ which contains u_ϵ. In turn this implies

$$u_\epsilon v_\epsilon=u_\epsilon v_\epsilon u_\epsilon=u_\epsilon;$$

and it follows that indeed $uv=u$. Thus u is an element of the minimal left ideal of $E(N,\mathcal{G}^{[d]})$ which contains v, an therefore u is a minimal idempotent of $E(N,\mathcal{G}^{[d]})$.

Now let x be an arbitrary point in A and let $u\in E(N,T^{[d]})$ be a minimal idempotent with $ux=x$. By the above argument, u is also a minimal idempotent of $E(N,\mathcal{G}^{[d]})$, whence $N=\overline{\mathcal{O}(A,\mathcal{F}^{[d]})}=\overline{\mathcal{O}(x,\mathcal{G}^{[d]})}$ is $\mathcal{G}^{[d]}$-minimal.

Finally, we show $\mathcal{F}^{[d]}$-minimal points are dense in N. Let $B\subseteq N$ be an $\mathcal{F}^{[d]}$-minimal subset. Then $\mathcal{O}(B,T^{[d]})=\bigcup\{(T^{[d]})^nB:n\in\mathbb{Z}\}$ is a $\mathcal{G}^{[d]}$-invariant subset of N. Since $(N,\mathcal{G}^{[d]})$ is minimal, $\mathcal{O}(B,T^{[d]})$ is dense in N. Note that every point in $\mathcal{O}(B,T^{[d]})$ is $\mathcal{F}^{[d]}$-minimal, hence the proof is completed. $\square$

Setting $A=\Delta^{[d]}$ we have

Corollary 4.9. *Let (X,T) be a minimal system and let $d\geq 1$ be an integer. Then $(\mathbf{Q}^{[d]},\mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in $\mathbf{Q}^{[d]}$.*

5. Proof of Theorem 3.1-(2)

In this section we prove Theorem 3.1-(2). That is, we show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}),\mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$ for all $x\in X$.

5.1. **A useful lemma.** The following lemma is a key step to show the uniqueness of minimal sets in $\mathbf{Q}^{[d]}[x]$ for $x\in X$. Unlike the case when (X,T) is minimal distal, we need to use the enveloping semigroup theory.

Lemma 5.1. *Let (X,T) be a minimal system and let $d\geq 1$ be an integer. If $(x^{[d-1]},\mathbf{w})\in\mathbf{Q}^{[d]}(X)$ for some $\mathbf{w}\in X^{[d-1]}$ and it is $\mathcal{F}^{[d]}$-minimal, then*

$$(x^{[d-1]},\mathbf{w})\in\overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

Proof. Since $(x^{[d-1]},\mathbf{w})\in\mathbf{Q}^{[d]}(X)$ and $(\mathbf{Q}^{[d]},\mathcal{G}^{[d]})$ is a minimal system by Corollary 4.9, $(x^{[d-1]},\mathbf{w})$ is in the $\mathcal{G}^{[d]}$-orbit closure of $x^{[d]}$, i.e. there are sequences $\{n_k\}_k,\{n_k^1\}_k,\ldots,\{n_k^d\}_k\subseteq\mathbb{Z}$ such that

$$(T_d^{[d]})^{n_k}(T_1^{[d]})^{n_k^1}\ldots(T_{d-1}^{[d]})^{n_k^{d-1}}(T^{[d]})^{n_k^d}(x^{[d-1]},x^{[d-1]})\to(x^{[d-1]},\mathbf{w}),\ k\to\infty.$$

Let

$$\mathbf{a_k}=(T_1^{[d-1]})^{n_k^1}\ldots(T_{d-1}^{[d-1]})^{n_k^{d-1}}(T^{[d-1]})^{n_k^d}(x^{[d-1]}),$$

then the above limit can be rewritten as

$$(T_d^{[d]})^{n_k}(\mathbf{a_k},\mathbf{a_k}) = (\mathrm{id}^{[d-1]}\times T^{[d-1]})^{n_k}(\mathbf{a_k},\mathbf{a_k}) \to (x^{[d-1]},\mathbf{w}),\ k\to\infty. \tag{5.1}$$

Let

$$\pi_1 : (X^{[d]},\mathcal{F}^{[d]}) \to (X^{[d-1]},\mathcal{F}^{[d]}), \quad (\mathbf{x}',\mathbf{x}'')\mapsto \mathbf{x}',$$
$$\pi_2 : (X^{[d]},\mathcal{F}^{[d]}) \to (X^{[d-1]},\mathcal{F}^{[d]}), \quad (\mathbf{x}',\mathbf{x}'')\mapsto \mathbf{x}'',$$

be projections to the first 2^{d-1} coordinates and last 2^{d-1} coordinates respectively. For π_1 we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]}\mapsto T_i^{[d-1]}$ for $1\le i\le d-1$ and $T_d^{[d]}\mapsto \mathrm{id}^{[d-1]}$. For π_2 we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]}\mapsto T_i^{[d-1]}$ for $1\le i\le d-1$ and $T_d^{[d]}\mapsto T^{[d-1]}$.

Denote the corresponding semigroup homomorphisms of enveloping semigroups by

$$\pi_1^* : E(X^{[d]},\mathcal{F}^{[d]}) \to E(X^{[d-1]},\mathcal{F}^{[d]}), \quad \pi_2^* : E(X^{[d]},\mathcal{F}^{[d]}) \to E(X^{[d-1]},\mathcal{F}^{[d]}).$$

Notice that for this action of $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ clearly

$$\pi_1^*(E(X^{[d]},\mathcal{F}^{[d]})) = E(X^{[d-1]},\mathcal{F}^{[d-1]}) \text{ and } \pi_2^*(E(X^{[d]},\mathcal{F}^{[d]})) = E(X^{[d-1]},\mathcal{G}^{[d-1]})$$

as subsets of $(X^{[d-1]})^{X^{[d-1]}}$. Thus for any $p\in E(X^{[d]},\mathcal{F}^{[d]})$ and $\mathbf{x}\in X^{[d]}$, we have

$$p\mathbf{x} = p(\mathbf{x}',\mathbf{x}'') = (\pi_1^*(p)\mathbf{x}',\pi_2^*(p)\mathbf{x}'').$$

Now fix a minimal left ideal $\mathbf{L}$ of $E(X^{[d]},\mathcal{F}^{[d]})$. By (5.1), $\mathbf{a_k}\to x^{[d-1]}, k\to\infty$. Since $(\mathbf{Q}^{[d-1]}(X),\mathcal{G}^{[d-1]})$ is minimal, there exists $p_k\in\mathbf{L}$ such that $\mathbf{a_k}=\pi_2^*(p_k)x^{[d-1]}$. Without loss of generality, we assume that $p_k\to p\in\mathbf{L}$. Then

$$\pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k}\to x^{[d-1]} \text{ and } \pi_2^*(p_k)x^{[d-1]}\to \pi_2^*(p)x^{[d-1]}.$$

Hence

$$\pi_2^*(p)x^{[d-1]} = x^{[d-1]}. \tag{5.2}$$

Since $\mathbf{L}$ is a minimal left ideal and $p\in\mathbf{L}$, by Proposition A.1 there exists a minimal idempotent $v\in J(\mathbf{L})$ such that $vp=p$. Then we have

$$\pi_2^*(v)x^{[d-1]} = \pi_2^*(v)\pi_2^*(p)x^{[d-1]} = \pi_2^*(vp)x^{[d-1]} = \pi_2^*(p)x^{[d-1]} = x^{[d-1]}.$$

Let

$$F = \mathfrak{G}(\overline{\mathcal{F}^{[d-1]}}(x^{[d-1]}),x^{[d-1]}) = \{\alpha\in v\mathbf{L} : \pi_2^*(\alpha)x^{[d-1]} = x^{[d-1]}\}$$

be the Ellis group. Then F is a subgroup of the group $v\mathbf{L}$. By (5.2), we have that $p\in F$.

Since F is a group and $p\in F$. We have

$$pFx^{[d]} = Fx^{[d]} \subset \pi_2^{-1}(x^{[d-1]}). \tag{5.3}$$

Since $vx^{[d]}\in Fx^{[d]}$, there is some $\mathbf{x_0}\in Fx^{[d]}$ such that $vx^{[d]}=p\mathbf{x_0}$. Set $\mathbf{x_k}=p_k\mathbf{x_0}$. Then

$$\mathbf{x_k} = p_k\mathbf{x_0}\to p\mathbf{x_0} = vx^{[d]} = (\pi_1^*(v)x^{[d-1]},x^{[d-1]}),\ k\to\infty,$$

and

$$\pi_2(\mathbf{x_k}) = \pi_2(p_k\mathbf{x_0}) = \pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k}\to x^{[d-1]},\ k\to\infty.$$

Let $\mathbf{x_k}=(\mathbf{b_k},\mathbf{a_k})\in\overline{\mathcal{F}^{[d]}}(x^{[d]})$. Then $\lim_k \mathbf{b_k} = \pi_1^*(v)x^{[d-1]}$.

By (5.1), we have $(T^{[d-1]})^{n_k}\mathbf{a_k} \to \mathbf{w}, k \to \infty$. Hence

$$(\mathrm{id}^{[d-1]} \times T^{[d-1]})^{n_k}(\mathbf{b_k}, \mathbf{a_k}) = (\mathbf{b_k}, (T^{[d-1]})^{n_k}\mathbf{a_k}) \to (\pi_1^*(v)x^{[d-1]}, \mathbf{w}),\ k \to \infty. \tag{5.4}$$

Since $\mathrm{id}^{[d-1]} \times T^{[d-1]} = T_d^{[d]} \in \mathcal{F}^{[d]}$ and $(\mathbf{b_k}, \mathbf{a_k}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have

$$(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}). \tag{5.5}$$

Since $(x^{[d-1]}, \mathbf{w})$ is $\mathcal{F}^{[d]}$ minimal by assumption, by Proposition A.2 there is some minimal idempotent $u \in J(\mathbf{L})$ such that

$$u(x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) = (x^{[d-1]}, \mathbf{w}).$$

Since $u, v \in \mathbf{L}$ are minimal idempotents in the same minimal left ideal $\mathbf{L}$, we have $uv = u$ by Proposition A.1. Thus

$$\begin{aligned} u(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) &= (\pi_1^*(u)\pi_1^*(v)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) \\ &= (\pi_1^*(uv)x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \mathbf{w}) = (x^{[d-1]}, \mathbf{w}). \end{aligned}$$

By (5.5), we have

$$(x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

The proof is completed. □

5.2. **Proof of Theorem 3.1-(2).** Let (X, T) be a system and $x \in X$. Recall

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

With the help of Lemma 5.1 we have

Proposition 5.2. *Let (X, T) be a minimal system and let $d \geq 1$ be an integer. If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then*

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Especially, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$.

Proof. It is sufficient to show the following claim:

S(d): *If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then there exists a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(\mathbf{x}) \to x^{[d]}$.*

The case $\mathbf{S(1)}$ is trivial. To make the idea clearer, we show the case when $d = 2$. Let $(x, a, b, c) \in \mathbf{Q}^{[2]}(X)$. We may assume that (x, a, b, c) is $\mathcal{F}^{[2]}$-minimal, or we replace it by some $\mathcal{F}^{[2]}$-minimal point in its $\mathcal{F}^{[2]}$ orbit closure. Since (X, T) is minimal, there is a sequence $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k}a \to x$. Without loss of generality we assume $T^{n_k}c \to c'$. Then we have

$$(T_1^{[2]})^{n_k}(x, a, b, c) = (\mathrm{id} \times T \times \mathrm{id} \times T)^{n_k}(x, a, b, c) \to (x, x, b, c'),\ k \to \infty.$$

Since (x, a, b, c) is $\mathcal{F}^{[2]}$-minimal, (x, x, b, c') is also $\mathcal{F}^{[2]}$-minimal. By Lemma 5.1, $(x, x, b, c') \in \overline{\mathcal{F}^{[2]}}(x^{[2]})$. Together with $\mathrm{id} \times T \times \mathrm{id} \times T = T_1^{[2]} \in \mathcal{F}^{[2]}$ and the minimality of the system $(\overline{\mathcal{F}^{[2]}}(x^{[2]}), \mathcal{F}^{[2]})$ (Theorem 3.1-(1)), it is easy to see there exists a sequence $F_k \in \mathcal{F}^{[2]}$ such that $F_k(x, a, b, c) \to x^{[2]}$. Hence we have $\mathbf{S(2)}$.

Now we assume $\mathbf{S(d)}$ holds for $d \geq 1$. Let $\mathbf{x} \in \mathbf{Q}^{[d+1]}[x]$. We may assume that $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, or we replace it by some $\mathcal{F}^{[d+1]}$-minimal point in its $\mathcal{F}^{[d+1]}$-orbit closure. Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}', \mathbf{x}'' \in X^{[d]}$. Then $\mathbf{x}' \in \mathbf{Q}^{[d]}[x]$. By $\mathbf{S(d)}$, there is

a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k \mathbf{x}' \to x^{[d]}$. Without loss of generality, we assume that $F_k \mathbf{x}'' \to \mathbf{w}, k \to \infty$. Then

$$(F_k \times F_k)\mathbf{x} = (F_k \times F_k)(\mathbf{x}', \mathbf{x}'') \to (x^{[d]}, \mathbf{w}) \in \mathbf{Q}^{[d+1]}(X),\ k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, $(x^{[d]}, \mathbf{w})$ is also $\mathcal{F}^{[d+1]}$-minimal. By Lemma 5.1, $(x^{[d]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. Since $(\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal by Theorem 3.1-(1), we have $x^{[d+1]}$ is in the $\mathcal{F}^{[d+1]}$-orbit closure of $\mathbf{x}$. Hence we have $\mathbf{S(d+1)}$, and the proof of claim is completed.

Since $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}))$ for all $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$ and $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal, it is easy to see that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ intersects all $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}[x]$ and hence it is the unique $\mathcal{F}^{[d]}$-minimal set in $\mathbf{Q}^{[d]}[x]$. The proof is completed. □

6. Lifting $\mathbf{RP}^{[d]}$ from factors to extensions

In this section, first we give some equivalent conditions for $\mathbf{RP}^{[d]}$, and give the proof of Theorem 3.8-(2), i.e. lifting $\mathbf{RP}^{[d]}$ from factors to extensions.

6.1. Equivalent conditions for $\mathbf{RP}^{[d]}$.

In this subsection we collect some equivalent conditions for $\mathbf{RP}^{[d]}$.

Proposition 6.1. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

1. $(x,y) \in \mathbf{RP}^{[d]}$;
2. $(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;
3. $(x, x_*^{[d]}, y, x_*^{[d]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

Proof. By Theorem 3.2, we have (1) $\Leftrightarrow$ (2). By Lemma 2.6 we have (3) $\Rightarrow$ (1). Now show (2) $\Rightarrow$ (3).

We show it by induction on d. When $d = 1$, it is easy to see that (2) and (3) are equivalent. Now assume that (2) $\Rightarrow$ (3) for $d-1$.

If (2) holds for d, then $(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. Thus $(x,y) \in \mathbf{RP}^{[d]}$ by Lemma 2.6. Since $(x,y) \in \mathbf{RP}^{[d]} \subset \mathbf{RP}^{[d-1]}$, $(x, y_*^{[d]}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$. By Theorem 3.1, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. There is some sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(x, y_*^{[d]}) \to x^{[d]},\ k \to \infty$. Then

$$F_k \times F_k(x, y_*^{[d]}, y, y_*^{[d]}) \to (x, x_*^{[d]}, y, x_*^{[d]}),\ k \to \infty.$$

Thus we have (3) for d. The proof is completed. □

Lemma 6.2. *Let (X,T) be a minimal system. Then $(x,y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$.*

Proof. If $(x,y) \in \mathbf{RP}^{[d]}$, then by Proposition 6.1, we have $(x, x_*^{[d]}, y, x_*^{[d]}) = (x^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$.

Conversely, assume that $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x, x_*^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. This means that $(x,y) \in \mathbf{RP}^{[d]}$ by Lemma 2.6. □

6.2. **Lifting $\mathbf{RP}^{[d]}$ from factors to extensions.** In this section we will show Theorem 3.8-(2). First we need a lemma.

Lemma 6.3. *Let $\pi : (X,T) \to (Y,T)$ be an extension of minimal systems. If $(y_1, y_2) \in \mathbf{P}(Y,T)$ and $x_1 \in \pi^{-1}(y_1)$ then there exists $x_2 \in \pi^{-1}(y_1)$ such that $(x_1, x_2) \in \mathbf{P}(X,T)$ and $\pi \times \pi(x_1, x_2) = (y_1, y_2)$.*

Proof. Since $(y_1, y_2) \in \mathbf{P}(Y,T)$, by Proposition A.3 there is an minimal idempotent $u \in E(X,T)$ such that $uy_1 = uy_2 = y_2$. Let $x_2 = ux_1$, then $\pi(x_2) = uy_1 = y_2$. By Proposition A.3 $(x_1, x_2) \in \mathbf{P}(X,T)$ and $\pi \times \pi(x_1, x_2) = (y_1, y_2)$. $\square$

Proposition 6.4. *Let $\pi : (X,T) \to (Y,T)$ be an extension of minimal systems. If $(y_1, y_2) \in \mathbf{RP}^{[d]}(Y)$, then there is $(z_1, z_2) \in \mathbf{RP}^{[d]}(X)$ such that*

$$\pi \times \pi(z_1, z_2) = (y_1, y_2).$$

Proof. First we claim that it is sufficient to show the result when (y_1, y_2) is a minimal point of $(Y \times Y, T \times T)$. As a matter of fact, by Proposition A.3 there is a minimal point $(y_1', y_2') \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$ such that (y_1', y_2') is proximal to (y_1, y_2). Now (y_1', y_2') is minimal and $(y_1', y_2') \in \mathbf{RP}^{[d]}(Y)$. If we have the claim already, then there is $(x_1', x_2') \in \mathbf{RP}^{[d]}(X)$ with $\pi \times \pi(x_1', x_2') = (y_1', y_2')$. Since $(y_1, y_1'), (y_2, y_2') \in \mathbf{P}(Y,T)$, then by Lemma 6.3 there are $x_1, x_2 \in X$ with $\pi \times \pi(x_1, x_2) = (y_1, y_2)$ such that (x_1', x_1), $(x_2', x_2) \in \mathbf{P}(X,T)$. This implies that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$ by Theorem 3.3. Hence we have the result for general case.

So we may assume that (y_1, y_2) is a minimal point of $(Y \times Y, T \times T)$. To make the idea of the proof clearer, we show the case for $d = 1$ first (see Figure 1). Since $(y_1, y_2) \in \mathbf{RP}^{[1]}(Y)$, by Proposition 6.1 $(y_1, y_1, y_2, y_1) \in \overline{\mathcal{F}^{[2]}}(y_1^{[2]})$. So there is some sequence $F_k \in \mathcal{F}^{[2]}$ such that

$$F_k y_1^{[2]} \to (y_1, y_1, y_2, y_1),\ k \to \infty.$$

Take a point $x_1 \in \pi^{-1}(y_1)$. Without loss of generality, we may assume that

$$F_k x_1^{[2]} \to (x_1, x_2, x_3, x_4),\ k \to \infty.$$

Then $\pi^{[2]}(x_1, x_2, x_3, x_4) = (y_1, y_1, y_2, y_1)$. Take $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k} x_2 \to x_1, k \to \infty$ and assume that $T^{n_k} x_4 \to x_4', k \to \infty$. Then

$$(\mathrm{id} \times T \times \mathrm{id} \times T)^{n_k}(x_1, x_2, x_3, x_4) \to (x_1, x_1, x_3, x_4'),\ k \to \infty.$$

Since $\mathrm{id} \times T \times \mathrm{id} \times T = T_1^{[2]} \in \mathcal{F}^{[2]}$, we have $(x_1, x_1, x_3, x_4') \in \overline{\mathcal{F}^{[2]}}(x_1^{[2]})$. Now take $\{m_k\} \subset \mathbb{Z}$ such that $T^{m_k} x_3 \to x_1, k \to \infty$ and assume that $T^{m_k} x_4' \to x_4'', k \to \infty$. Then

$$(\mathrm{id} \times \mathrm{id} \times T \times T)^{m_k}(x_1, x_1, x_3, x_4') \to (x_1, x_1, x_1, x_4''),\ k \to \infty.$$

Since $\mathrm{id} \times \mathrm{id} \times T \times T = T_2^{[2]} \in \mathcal{F}^{[2]}$, we have $(x_1, x_1, x_1, x_4'') \in \overline{\mathcal{F}^{[2]}}(x_1^{[2]})$. By Lemma 6.2 $(x_1, x_4'') \in \mathbf{RP}^{[1]}(X)$. Let $y_3 = \pi(x_4'')$. Note that $(x_1, x_4'') \in \overline{\mathcal{O}((x_3, x_4'), T \times T)}$, and we have $(y_3, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Since (y_1, y_2) is $T \times T$-minimal, there is a sequence $\{a_k\} \subset \mathbb{Z}$ such that $(T \times T)^{a_k}(y_3, y_1) \to (y_1, y_2), k \to \infty$. Without loss of generality, we may assume that there are $z_1, z_2 \in X$ such that

$$(T \times T)^{a_k}(x_4'', x_1) \to (z_1, z_2),\ k \to \infty$$

Since $(x_1, x_4'') \in \mathbf{RP}^{[1]}(X)$ and $\mathbf{RP}^{[1]}(X)$ is closed and invariant, we have $(z_1, z_2) \in \overline{\mathcal{O}((x_4'', x_1), T \times T)} \subset \mathbf{RP}^{[1]}(X)$. Note that

$$\pi \times \pi(z_1, z_2) = \lim_k (T \times T)^{a_k}(\pi(x_4''), \pi(x_1)) = \lim_k (T \times T)^{a_k}(y_3, y_1) = (y_1, y_2),$$

we are done for the case $d = 1$. For the proof when $d = 2$, see Figure 2.

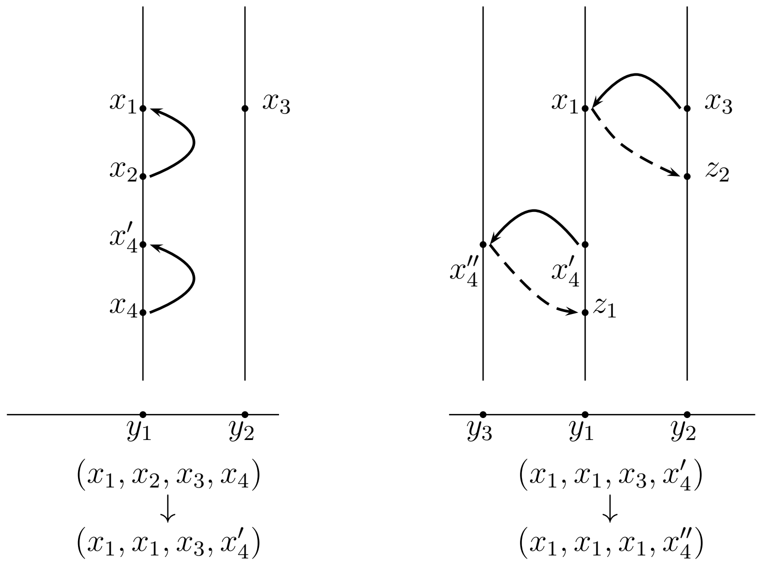

Figure 1. The case $d = 1$

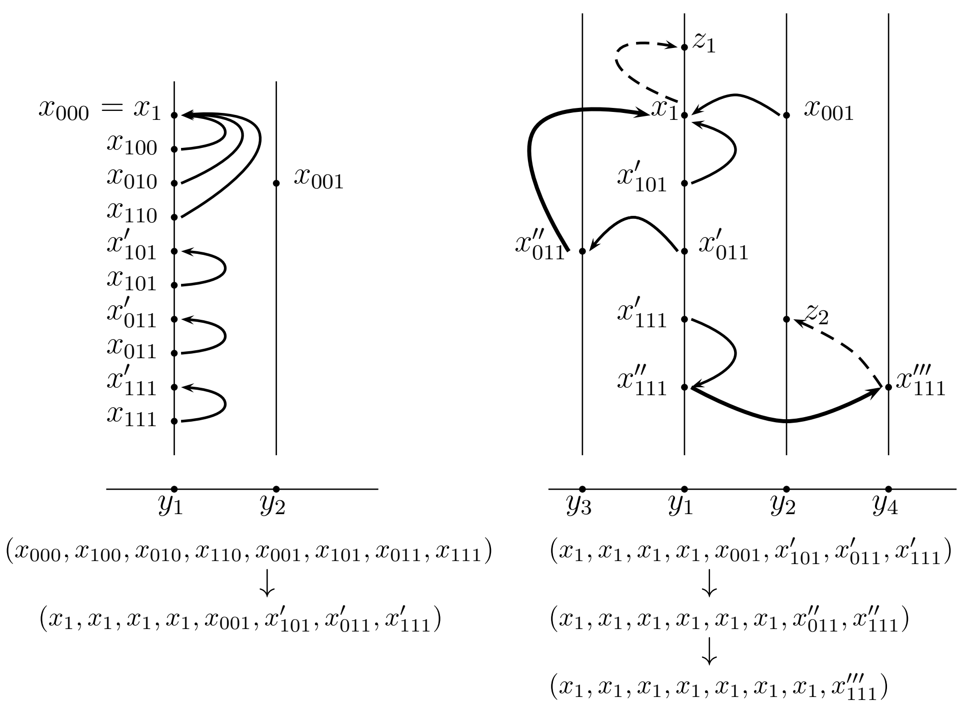

Figure 2. The case $d = 2$

The idea of the proof in the general case is the following. For a point $\mathbf{x} \in \overline{\mathcal{F}^{[d+1]}}(x_1)$ we apply face transformations F_1^k such that the first 2^d-coordinates of $\mathbf{x}_1 = \lim F_1^k \mathbf{x}$ will be $x_1^{[d]}$. Then apply face transformations F_2^k such that the first $2^d + 2^{d-1}$-coordinates of $\mathbf{x}_2 = \lim F_2 \mathbf{x}_1$ will be $(x_1^{[d]}, x_1^{[d-1]})$. Repeating this process we get a point $((x_1^{[d+1]})_*, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1)$ which implies that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$. Then we use the same idea used in the proof when $d = 1, 2$ to trace back to find (z_1, z_2). Here are the details.

Now let $(y_1, y_2) \in \mathbf{RP}^{[d]}(Y)$, then by Proposition 6.1, $(y_1^{[d]}, y_2, (y_1^{[d]})_*) \in \overline{\mathcal{F}^{[d+1]}}(y_1^{[d+1]})$. So there is some sequence $F_k \in \mathcal{F}^{[d+1]}$ such that

$$F_k y_1^{[d+1]} \to (y_1^{[d]}, y_2, (y_1^{[d]})_*), \ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k x_1^{[d+1]} \to \mathbf{x}, \ k \to \infty. \tag{6.1}$$

Then $x_\emptyset = x_1$ and $\pi^{[d+1]}(\mathbf{x}) = (y_1^{[d]}, y_2, (y_1^{[d]})_*)$.

Let $\mathbf{x_I} = (x_\epsilon : \epsilon(d+1) = 0) \in X^{[d]}$ and $\mathbf{x_{II}} = (x_\epsilon : \epsilon(d+1) = 1) \in X^{[d]}$. Then $\mathbf{x} = (\mathbf{x_I}, \mathbf{x_{II}})$. Note that

$$\pi^{[d]}(\mathbf{x_I}) = \pi^{[d]}(x_1^{[d]}) = y_1^{[d]}, \text{ and } \pi^{[d]}(\mathbf{x_{II}}) = (y_2, (y_1^{[d]})_*).$$

By Proposition 5.2, there is some sequence $F_k^1 \in \mathcal{F}^{[d]}$ such that

$$F_k^1(\mathbf{x_I}) \to x_1^{[d]}, \ k \to \infty.$$

We may assume that

$$F_k^1(\mathbf{x_{II}}) \to \mathbf{x'_{II}}, \ k \to \infty.$$

Note that $\pi^{[d]}(\mathbf{x_{II}}) = \pi^{[d]}(\mathbf{x'_{II}}) = (y_2, (y_1^{[d]})_*)$.

Let $F_k^1 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$. Let $H_k^1 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 1) = F_k^1.$$

Then

$$H_k^1(\mathbf{x}) = F_k^1 \times F_k^1(\mathbf{x_I}, \mathbf{x_{II}}) \to (x_1^{[d]}, \mathbf{x'_{II}}) \triangleq \mathbf{x^1} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}), \ k \to \infty.$$

Let $\mathbf{y^1} = \pi^{[d+1]}(\mathbf{x^1})$. It is easy to see that $x_\epsilon^1 = x_1$ if $\epsilon(d+1) = 0$. For $\mathbf{y^1}$, $y^1_{\{d+1\}} = y^1_{00\ldots 01} = y_2$ and $y^1_\epsilon = y_1$ for all $\epsilon \neq \{d+1\}$.

Let $\mathbf{x_I^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) \in X^{[d]}$ and $\mathbf{x_{II}^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F_k^2 \in \mathcal{F}^{[d]}$ such that

$$F_k^2(\mathbf{x_I^1}) \to x_1^{[d]}, \ F_k^2(\mathbf{x_{II}^1}) \to \mathbf{x_{II}^1}', k \to \infty$$

and $\pi^{[d]}(\mathbf{x_{II}^1}') = (y_1^{[d-1]}, y_3, (y_1^{[d-1]})_*)$ for some $y_3 \in Y$.

Let $F_k^2 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$. Let $H_k^2 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) = F_k^2.$$

Then let

$$H_k^2(\mathbf{x^1}) \to \mathbf{x^2} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}), \ k \to \infty.$$

Let $\mathbf{y^2} = \pi^{[d+1]}(\mathbf{x^2})$. Then $H_k^2(\mathbf{y^1}) \to \mathbf{y^2},\ k \to \infty$. From this one has that $(y_3, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. By the definition of $\mathbf{x^2}, \mathbf{y^2}$, it is easy to see that $x^2_\epsilon = x_1$ if $\epsilon(d+1) = 0$ or $\epsilon(d) = 0$; $y^2_{\{d,d+1\}} = y^2_{00\ldots011} = y_3$ and $y^2_\epsilon = y_1$ for all $\epsilon \neq \{d, d+1\}$.

Now assume that we have $\mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ for $1 \le j \le d$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$ such that $x^j_\epsilon = x_1$ if there exists some k with $d - j + 2 \le k \le d+1$ such that $\epsilon(k) = 0$; $y^j_{\{d-j+2,\ldots,d,d+1\}} = y_{j+1}$ and $y^j_\epsilon = y_1$ for all $\epsilon \neq \{d-j+2, \ldots, d, d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$.

Let $\mathbf{x^j_I} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) \in X^{[d]}$ and $\mathbf{x^j_{II}} = (x_\epsilon : \epsilon \in \{0,1\}^{d-j+1}, \epsilon(d-j+1) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F_k^{j+1} \in \mathcal{F}^{[d]}$ such that

$$F_k^{j+1}(\mathbf{x^j_I}) \to x_1^{[d]},\ F_k^{j+1}(\mathbf{x^j_{II}}) \to \mathbf{x^j_{II}}', k \to \infty.$$

Let $F_k^{j+1} = (S^k_{\epsilon'} : \epsilon' \in \{0,1\}^d)$. Let $H_k^{j+1} = (S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that $(S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) = (S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 1) = F_k^{j+1}$. Then let

$$H_k^{j+1}(\mathbf{x^j}) \to \mathbf{x^{j+1}} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}),\ k \to \infty.$$

It is easy to see that $x^{j+1}_\epsilon = x_1$ if there exists some k with $d - j + 1 \le k \le d+1$ such that $\epsilon(k) = 0$.

Let $\mathbf{y^{j+1}} = \pi^{[d+1]}(\mathbf{x^{j+1}})$. Then $y^{j+1}_\epsilon = y_1$ for all $\epsilon \neq \{d-j+1, d-j+2, \ldots, d+1\}$, and denote $y^j_{\{d-j+1,d-j+2,\ldots,d+1\}} = y_{j+2}$. Note that $H_k^2(\mathbf{y^j}) \to \mathbf{y^{j+1}},\ k \to \infty$. From this one has that $(y_{j+2}, y_1) \in \overline{\mathcal{O}((y_1, y_{j+1}), T \times T)}$.

Inductively we get $\mathbf{x^1}, \ldots, \mathbf{x^{d+1}}$ and $\mathbf{y^1}, \ldots, \mathbf{y^{d+1}}$ such that for all $1 \le j \le d+1$ $\mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$. And $x^j_\epsilon = x_1$ if there exists some k with $d - j + 2 \le k \le d+1$ such that $\epsilon(k) = 0$; $y^j_{\{d-j+2,\ldots,d,d+1\}} = y_{j+1}$ and $y^j_\epsilon = y_1$ for all $\epsilon \neq \{d-j+2, \ldots, d, d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$.

For $\mathbf{x^{d+1}}$, we have that $x^{d+1}_\epsilon = x_1$ if there exists some k with $1 \le k \le d+1$ such that $\epsilon(k) = 0$. That means there is some $x_2 \in X$ such that

$$\mathbf{x^{d+1}} = (x_1, x_1, \ldots, x_1, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}).$$

By Lemma 6.2, $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$. Note that $\pi(x_2) = y_{d+2}$.

Since $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1, y_j), T \times T)}$ for all $1 \le j \le d+1$, we have $(y_{d+2}, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$ or $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Without loss of generality, we assume that $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. Since (y_1, y_2) is $T \times T$-minimal, there is some $\{n_k\} \subset \mathbb{Z}$ such that $(T \times T)^{n_k}(y_1, y_{d+2}) \to (y_1, y_2), k \to \infty$. Without loss of generality, we assume that

$$(T \times T)^{n_k}(x_1, x_2) \to (z_1, z_2),\ k \to \infty.$$

Since $\mathbf{RP}^{[d]}(X)$ is closed and invariant, we have

$$(z_1, z_2) \in \overline{\mathcal{O}((x_1, x_2), T \times T)} \subset \mathbf{RP}^{[d]}(X).$$

And

$$\pi \times \pi(z_1, z_2) = \lim_k (T \times T)^{n_k}(\pi(x_1), \pi(x_2)) = \lim_k (T \times T)^{n_k}(y_1, y_{d+2}) = (y_1, y_2).$$

The proof is completed. $\square$

7. A combinatorial consequence and group actions

7.1. **A combinatorial consequence.** We have the following combinatorial consequence of the fact that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal.

Proposition 7.1. *Let (X,T) be a minimal system, $x \in X$ and U be an open neighborhood of x. Put $S = \{n \in \mathbb{Z} : T^n x \in U\}$. Then for each $d \geq 1$,*

$$\{(n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic.

Proof. This follows by that fact that $x^{[d]}$ is a minimal point under the face action $\mathcal{F}^{[d]}$. $\square$

To understand S better we show the following proposition which is similar to [24, Proposition 2.3]. Note that a collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ is a *family* if it is upwards, i.e. $A \in \mathcal{F}$ and $A \subset B$ imply that $B \in \mathcal{F}$.

Proposition 7.2. *The family of dynamically syndetic subsets is the family generated by the sets S whose indicator functions 1_S are the minimal points of $(\{0,1\}^{\mathbb{Z}}, \sigma)$ and $0 \in S$, where σ is the shift.*

Proof. Put $\Sigma = \{0,1\}^{\mathbb{Z}}$. We denote the family generated by the sets containing $\{0\}$ whose indicator functions are the minimal points of (Σ, σ) by $\mathcal{F}_m$. Clearly, if 1_F is the indicator function of F then $F = N(1_F, [1])$, where $[1] = \{s \in \Sigma : s(0) = 1\}$. Hence $\mathcal{F}_m$ is contained in the family of dynamical syndetic subsets.

On the other hand, let A be a dynamical syndetic subset. Then there exist a minimal system (X,T) with metric d, $x \in X$ and an open neighborhood V of x such that $A \supset N(x,V) = \{n \in \mathbb{Z} : T^n x \in V\}$. It is easy to see that we can shrink V to an open neighborhood V' of x whose boundary is disjoint from the orbit of x.

Then do the classical lifting trick, a la Glasner, Adler etc. Let

$$Y = \{(z,t) \in X \times \Sigma : t(i) = 1 \text{ implies } T^i z \in \mathrm{cl}(V') \text{ and } t(i) = 0 \text{ implies } T^i z \in \mathrm{cl}(X \setminus V')\}$$

Then Y is a $T \times \sigma$-invariant closed subset of $X \times \Sigma$. Since the orbit of x doesn't meet the boundary of V', there is a unique $t \in \Sigma$ such that $(x,t) \in Y$ and t is the indicator function of $N(x,V')$. Take a minimal subset J of $(Y, T \times \sigma)$ with $J \subset \overline{\mathcal{O}((x,t), T \times \sigma)}$ and let $\pi_X : J \to X$ be the projective map. Since (X,T) is minimal, $\pi_X(J) = X$. Hence $(x,t) \in J$. Projecting J to Σ we see that t is a minimal point. Hence $A \in \mathcal{F}_m$ as $A \supset N(x,V')$ and $t = 1_{N(x,V')}$. $\square$

Remark 7.3. We note that if S is a syndetic subset of $\mathbb{Z}$ then $S - S \supset S_1 - S_1$ for some dynamically syndetic subset S_1.

7.2. **Group actions.** Let X be a compact metric space and G be an abelian group.

Definition 7.4. Let X be a compact metric space, G be an abelian group actiong on X and let $d \geq 1$ be an integer. A pair $(x,y) \in X \times X$ ia said to be *regionally*

proximal of order d of G-action if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \ldots, n_d) \in G^d$ such that $d(x, x') < \delta, d(y, y') < \delta$, and

$$d(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \text{ for any nonempty } \epsilon \subset [d],$$

where $\mathbf{n} \cdot \epsilon = \sum_{i\in\epsilon} n_i$. The set of regionally proximal pairs of order d of G-action is denoted by $\mathbf{RP}_G^{[d]}(X)$, which is called *the regionally proximal relation of order d of G-action.*

A subset $S \subseteq G$ is a *central set* if there exists a system (X, G), a point $x \in X$ and a minimal point y proximal to x, and a neighborhood U_y of y such that $N(x, U_y) \subset S$. The notion of IP-set can be defined in this setting too. By the proof of Furstenberg [11, Proposition 8.10.] we have

Lemma 7.5. *Let G be an abelian group. Then any central set is an IP-set.*

So we have

Lemma 7.6. *If (X, G) is minimal, then $\mathbf{P}(X) \subset \mathbf{RP}_G^{[d]}(X)$.*

At the same time the notions of face group and parallelepiped group can be defined. So we have the following theorem by our proof

Theorem 7.7. *Let (X, G) a minimal system with G being abelian. Then $\mathbf{RP}_G^{[d]}(X)$ is a closed invariant equivalence relation. So $(X/\mathbf{RP}_G^{[d]}(X), G)$ is distal.*

Similar to [22] we may define

Definition 7.8. Let (X, G) a minimal system with G being abelian. We call $(X/\mathbf{RP}_G^{[d]}(X), G)$ the d-step nilsystem for G-action.

We think that to study the properties of $(X/\mathbf{RP}_G^{[d]}(X), G)$ or more general group actions will be interesting.

Appendix A. Basic facts about abstract topological dynamics

In this section we recall some basic definitions and results in abstract topological systems. For more details, see [1, 4, 13, 16, 28, 29].

A.1. **Topological transformation groups.** A *topological dynamical systems* is a triple $\mathcal{X} = (X, \mathcal{T}, \Pi)$, where X is a compact T_2 space, $\mathcal{T}$ is a T_2 topological group and $\Pi : T \times X \to X$ is a continuous map such that $\Pi(e, x) = x$ and $\Pi(s, \Pi(t, x)) = \Pi(st, x)$. We shall fix $\mathcal{T}$ and suppress the action symbol. In lots of literatures, $\mathcal{X}$ is also called a *topological transformation group* or a *flow*.

Let $(X, \mathcal{T})$ be a system and $x \in X$, then $\mathcal{O}(x, \mathcal{T})$ denotes the *orbit* of x, which is also denoted by $\mathcal{T}x$. A subset $A \subseteq X$ is called *invariant* if $ta \subseteq A$ for all $a \in A$ and $t \in \mathcal{T}$. When $Y \subseteq X$ is a closed and $\mathcal{T}$-invariant subset of the system $(X, \mathcal{T})$ we say that the system $(Y, \mathcal{T})$ is a *subsystem* of $(X, \mathcal{T})$. If $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ are two dynamical systems their *product system* is the system $(X \times Y, \mathcal{T})$, where $t(x, y) = (tx, ty)$.

A system $(X, \mathcal{T})$ is called *minimal* if X contains no proper closed invariant subsets. $(X, \mathcal{T})$ is called *transitive* if every invariant open subset of X is dense. An example

of an transitive system is a *point-transitive* system, which is a system with a dense orbit. It is easy to verify that a system is minimal iff every orbit is dense. The system $(X, \mathcal{T})$ is *weakly mixing* if the product system $(X \times X, \mathcal{T})$ is transitive.

A *homomorphism* (or *extension*) of systems $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is a continuous onto map of the phase spaces such that $\pi(tx) = t\pi(x)$ for all $t \in \mathcal{T}, x \in X$. In this case one says that $(Y, \mathcal{T})$ if a *factor* of $(X, \mathcal{T})$ and also that $(X, \mathcal{T})$ is an *extension* of $(Y, \mathcal{T})$. Define

$$R_\pi = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\},$$

then $Y = X/R_\pi$. For $n \in \mathbb{N}$, define

$$R_\pi^n = \{(x_1, x_2, \dots, x_n) : \pi(x_1) = \pi(x_2) = \dots = \pi(x_n)\},$$

A.2. **Enveloping semigroups.** Given a system $(X, \mathcal{T})$ its *enveloping semigroup* or *Ellis semigroup* $E(X, \mathcal{T})$ is defined as the closure of the set $\{t : t \in \mathcal{T}\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). For an enveloping semigroup, $E \to E : p \mapsto pq$ and $p \mapsto tp$ is continuous for all $q \in E$ and $t \in \mathcal{T}$. Note that $(X^X, \mathcal{T})$ is a system and $(E(X, \mathcal{T}), \mathcal{T})$ is its subsystem.

Let $(X, \mathcal{T}), (Y, \mathcal{T})$ be systems and $\pi : X \to Y$ be an extension. Then there is a unique continuous semigroup homomorphism $\pi^* : E(X, \mathcal{T}) \to E(Y, \mathcal{T})$ such that $\pi(px) = \pi^*(p)\pi(x)$ for all $x \in X, p \in E(X, \mathcal{T})$. When there is no confusion, we usually regard the enveloping semigroup of X as acting on Y: $p\pi(x) = \pi(px)$ for $x \in X$ and $p \in E(X, \mathcal{T})$.

A.3. **Idempotents and ideals.** For a semigroup the element u with $u^2 = u$ is called an *idempotent*. Ellis-Namakura Theorem says that for any enveloping semigroup E the set $J(E)$ of idempotents of E is not empty [4]. A non-empty subset $I \subset E$ is a *left ideal* (resp. *right ideal*) if it $EI \subseteq I$ (resp. $IE \subseteq I$). A *minimal left ideal* is the left ideal that does not contain any proper left ideal of E. Obviously every left ideal is a semigroup and every left ideal contains some minimal left ideal.

We can introduce a quasi-order (a reflexive, transitive relation) $<_L$ on the set $J(E)$ by defining $v <_L u$ if and only if $vu = v$. If $v <_L u$ and $u <_L v$ we say that u and v are *equivalent* and write $u \sim_L v$. Similarly, we define $<_R$ and $\sim_R$. An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $v <_L u$ implies $u <_L v$. The following results are well-known [5, 12]: let L be a left ideal of enveloping semigroup E and $u \in J(E)$. Then there is some idempotent v in Lu such that $v <_R u$ and $v <_L u$; an idempotent is minimal if and only if it is contained in some minimal left ideal.

Minimal left ideals have very rich algebraic properties. For example,

Proposition A.1. *Let I be a minimal left ideal, then*

(1) $I = \bigcup_{u \in J(I)} uI$ *is its partition and every uI is a group with identity $u \in J(I)$.*
(2) *All minimal idempotents in the same left ideal are equivalent to each other, i.e. for all $u, v \in J(I)$, $u \sim_L v$.*

Let $(X,\mathcal{T})$ be a system and $E(X,\mathcal{T})$ be its enveloping semigroup. A subset $I\subseteq E(X,\mathcal{T})$ is a closed left ideal of $E(X,\mathcal{T})$ iff $(I,\mathcal{T})$ is a subsystem of $(E(X,\mathcal{T}),\mathcal{T})$. And I is a minimal left ideal of $E(X,\mathcal{T})$ iff $(I,\mathcal{T})$ is minimal. Let $I\subset E(X,\mathcal{T})$ be a minimal left ideal. Then for all $x\in X$, $Ix=\{px:p\in I\}$ is a minimal subset of X. Especially if $(X,\mathcal{T})$ is minimal itself, then $X=Ix$ for all $x\in X$. It follows that

Proposition A.2. *A point $x\in X$ is minimal if and only if $ux=x$ for some $u\in I$.*

A.4. **Universal point transitive system and universal minimal system.** For fixed $\mathcal{T}$, there exists a universal point-transitive system $\mathcal{S}_{\mathcal{T}}=(S_{\mathcal{T}},\mathcal{T})$ such that $\mathcal{T}$ can densely and equivariantly be embedded in $S_{\mathcal{T}}$. The multiplication on $\mathcal{T}$ can be extended to a multiplication on $S_{\mathcal{T}}$, then $S_{\mathcal{T}}$ is a closed semigroup with continuous right translations. The universal minimal system $\mathfrak{M}=(\mathbf{M},\mathcal{T})$ is isomorphic to any minimal left ideal in $S_{\mathcal{T}}$ and $\mathbf{M}$ is a closed semigroup with continuous right translations. Hence $J=J(\mathbf{M})$ of idempotents in $\mathbf{M}$ is nonempty. Moreover, $\{v\mathbf{M}:v\in J\}$ is a partition of $\mathbf{M}$ and every $v\mathbf{M}$ is a group with unit element v. Sometimes if there are chances being confusion then we will use $\mathbf{M}_{\mathcal{T}}$ instead of $\mathbf{M}$.

The sets $S_{\mathcal{T}}$ and $\mathbf{M}$ act on X as semigroups and $S_{\mathcal{T}}x=\overline{\mathcal{T}x}$, while for a minimal system $(X,\mathcal{T})$ we have $\mathbf{M}x=\overline{\mathcal{T}x}=X$ for every $x\in X$. A necessary and sufficient condition for x to be minimal is that $ux=x$ for some $u\in J$.

A.5. **All kinds of extensions.** Two points x_1 and x_2 are called *proximal* iff

$$\overline{\mathcal{T}(x_1,x_2)}\cap\Delta_X\neq\emptyset.$$

Let $\mathcal{U}_X$ be the unique uniform structure of X, then

$$\mathbf{P}(X,\mathcal{T})=\bigcap\{\mathcal{T}\alpha:\alpha\in\mathcal{U}_X\}$$

is the collection of proximal pairs in X, the *proximal relation*.

Proposition A.3. *Let $(X,\mathcal{T})$ be a system. Then*

(1) *x_1, x_2 are proximal in $(X,\mathcal{T})$ iff $px_1=px_2$ for some $p\in E(X,\mathcal{T})$.*
(2) *If $x\in X$ and u is an idempotent in $E(X,\mathcal{T})$, then $(x,ux)\in\mathbf{P}$.*
(3) *If $x\in X$, then there is an minimal point $x'\in\overline{\mathcal{O}(x,\mathcal{T})}$ such that $(x,x')\in\mathbf{P}$.*
(4) *If (X,T) is minimal, then $(x,y)\in\mathbf{P}$ if and only if there is some minimal idempotent $u\in E(X,\mathcal{T})$ such that $y=ux$.*

The extension $\pi:(X,\mathcal{T})\to(Y,\mathcal{T})$ is called *proximal* iff $R_\pi\subseteq\mathbf{P}$ iff $\mathbf{P}_\pi=\bigcap\{\mathcal{T}\alpha\cap R_\pi:\alpha\in\mathcal{U}_X\}=R_\pi$. π is *distal* if $\mathbf{P}_\pi=\Delta_X$. π is a *highly proximal* (HP) extension if for every closed subset A of X with $\pi(A)=Y$, necessarily $A=X$. It is easy to see that a HP extension is proximal. In the metric case an extension $\pi:(X,T)\to(Y,T)$ of minimal systems is HP iff it is an *almost 1-1 extension*, that is the set $\{y\in Y:\pi^{-1}(y)$ is a singleton $\}$ is a dense G_δ subset of Y.

An extension $\pi:X\to Y$ of systems is called *equicontinuous* or *almost periodic* if for every $\alpha\in\mathcal{U}_X$ there is $\beta\in\mathcal{U}_X$ such that $\mathcal{T}\alpha\cap R_\pi\subseteq\beta$.

In the metric case an equicontinuous extension is also called an *isometric extension.* The extension π is a *weakly mixing extension* when $(R_\pi,\mathcal{T})$ as a subsystem of the product system $(X\times X,\mathcal{T})$ is transitive.

A.6. **Vietoris topology and circle operation.** Let 2^X be the collection of nonempty closed subsets of X endowed with the Vietoris topology. Note that a base for the Vietoris topology on 2^X is formed by the sets

$$< U_1, U_2, \cdots, U_n >= \{A \in 2^X : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for every } i\},$$

where U_i is open in X. Then $(2^X, \mathcal{T})$ defined by $tA = \{ta : a \in A\}$ is a system again, and $S_{\mathcal{T}}$ acts on 2^X too. To avoid ambiguity we denote the action of $S_{\mathcal{T}}$ on 2^X by the *circle operation* as follows. Let $p \in S_{\mathcal{T}}$ and $D \in 2^X$, then define $p \circ D = \lim_{2^X} t_i D$ for any net $\{t_i\}_i$ in $\mathcal{T}$ with $t_i \to p$. Moreover

$$p \circ D = \{x \in X : \text{there are } d_i \in D \text{ with } x = \lim_i t_i d_i\}$$

for any net $t_i \to p$ in $S_{\mathcal{T}}$. We always have $pD \subseteq p \circ D$.

A.7. **Ellis group.** The group of automorphisms of $(\mathbf{M}, \mathcal{T})$, $G = \mathrm{Aut}(\mathbf{M}, \mathcal{T})$ can be identified with any one of the groups $u\mathbf{M}$ $(u \in J)$ as follows: with $\alpha \in uM$ we associate the automorphism $\hat{\alpha} : (\mathbf{M}, \mathcal{T}) \to (\mathbf{M}, \mathcal{T})$ given by right multiplication $\hat{\alpha}(p) = p\alpha, p \in \mathbf{M}$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology, called by Ellis the *τ-topology*, which is weaker than the relative topology induced on $G = u\mathbf{M}$ as a subset of $\mathbf{M}$.

It is convenient to fix a minimal left ideal $\mathbf{M}$ in $S_{\mathcal{T}}$ and an idempotent $u \in \mathbf{M}$. As explained above we identify G with $u\mathbf{M}$ and for any subset $A \subseteq G$, *τ-topology* is determined by

$$\mathrm{cl}_\tau A = u(u \circ A) = G \cap (u \circ A).$$

Also in this way we can consider the "action" of G on every system $(X, \mathcal{T})$ via the action of $S_{\mathcal{T}}$ on X. With every minimal system (X, T) and a point $x_0 \in uX = \{x \in X : ux = x\}$ we associate a τ-closed subgroup

$$\mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$$

the *Ellis group* of the pointed system (X, x_0).

For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathfrak{G}(X, x_0) \subseteq \mathfrak{G}(Y, y_0).$$

It is easy to see that $u\pi^{-1}(y_0) = \mathfrak{G}(Y, y_0)x_0$.

For a τ-closed subgroup F of G the derived group $H(F) = F'$ is given by:

$$H(F) = F' = \bigcap \big\{\mathrm{cl}_\tau O : O \text{ is a } \tau\text{-open neighborhood of } u \text{ in } F\ \big\}.$$

$H(F)$ is a τ-closed normal subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group. In particular, for an abelian $\mathcal{T}$, the topological group $G/H(G)$ is the *Bohr compactification* of $\mathcal{T}$.

A.8. **Structure of minimal systems.** We say that $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is a *RIC* (relatively incontractible) extension if for every $y = py_0 \in Y$, p an element of $\mathbf{M}$,

$$\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0,$$

where $F = \mathfrak{G}(Y, y_0)$. One can show that the extension $\pi : X \to Y$ is RIC if and only if it is open and for every $n \geq 1$ the minimal points are dense in the relation R^n_π. Note that every distal extension is RIC. It then follows that every distal extension is open.

We say that a minimal system $(X, \mathcal{T})$ is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that $(X, \mathcal{T})$ is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost one-to-one extensions (or by highly proximal extensions in the non-metric case) we get the notion of HPI *systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have I *systems*. These notions can be easily relativized and we then speak about I, HPI, and PI extensions.

Theorem A.4 (Furstenberg)**.** *A metric minimal system is distal if and only if it is an I-system.*

Theorem A.5 (Veech)**.** *A metric minimal dynamical system is point distal if and only if it is an HPI-system.*

Finally we have the structure theorem for minimal systems, which we will state in its relative form (Ellis-Glasner-Shapiro [7], Veech [28], and Glasner [13]).

Theorem A.6 (Structure theorem for minimal systems)**.** *Given a homomorphism $\pi : X \to Y$ of minimal dynamical system, there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{tikzcd}
X \arrow[d, "\pi"'] & X_0 \arrow[l, "\theta_0^*"'] \arrow[d, "\pi_0"'] \arrow[dr, "\sigma_1"] & & X_1 \arrow[ll, "\theta_1^*"'] \arrow[d, "\pi_1"'] & \cdots & X_\nu \arrow[d, "\pi_\nu"'] \arrow[dr, "\sigma_{\nu+1}"] & & X_{\nu+1} \arrow[ll, "\theta_{\nu+1}^*"'] \arrow[d, "\pi_{\nu+1}"'] & \cdots & X_\eta = X_\infty \arrow[d, "\pi_\infty"'] \\
Y & Y_0 \arrow[l, "\theta_0"] & Z_1 \arrow[l, "\rho_1"] & Y_1 \arrow[l, "\theta_1"] & \cdots & Y_\nu & Z_{\nu+1} \arrow[l, "\rho_{\nu+1}"] & Y_{\nu+1} \arrow[l, "\theta_{\nu+1}"] & \cdots & Y_\eta = Y_\infty
\end{tikzcd}$$

where for each $\nu \leq \eta, \pi_\nu$ is RIC, ρ_ν is isometric, θ_ν, θ^*_ν are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI-system.

Appendix B. Proof of Theorem 4.3

First we need the so-called *Ellis trick* in [13]. Refer to [13, Lemma X.6.1] for the proof. See [17] for more discussions about weakly mixing extensions. Recall that $\mathbf{M}$ is the universal minimal set.

Lemma B.1 (Ellis trick)**.** *Let F be τ closed subgroup of G acting on $\mathbf{M}$ by right multiplication, $\mathbf{M} \times F \to \mathbf{M}, (p, \alpha) \mapsto p\alpha$.*

(1) *there is a minimal idempotent $\omega \in J(\mathbf{M}) \cap \overline{F}$ such that $\overline{\omega F}$ is F-minimal.*
(2) *if V is a open subset of $\overline{wF}$, then $\mathrm{int}_\tau \mathrm{cl}_\tau(V \cap wF) \neq \emptyset$.*

Lemma B.2. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems and $u \in J(\mathbf{M})$ be a minimal idempotent. Let $x \in uX$, $y = \pi(x)$. Then for all $n \geq 1$, any nonempty open subset U of $\overline{u\pi^{-1}(y)}$ and any transitive point $x' = (x'_1, \cdots, x'_{n-1}) \in R_\pi^{n-1}$ with $\pi(x'_j) = y, j = 1, \cdots, n-1$, we have $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$.*

Proof. Note that we have $H(F)A = F$, where $F = \mathfrak{G}(Y, y), A = \mathfrak{G}(X, x)$, since π is weakly mixing.
Claim:

$$\{ux'\} \times \pi^{-1}(y) \subset \overline{\mathcal{T}(\{x'\} \times U)}.$$

Proof of The Claim: Set $V = \{p \in \overline{F} : px \in U\}$. Then V is a nonempty open set of $\overline{F}$ and by Ellis trick we have $\widetilde{V} = \mathrm{int}_\tau \mathrm{cl}_\tau(V \cap F) \neq \emptyset$. By the definition of $H(F)$, there exists $\alpha \in F$ such that $\alpha H(F) \subseteq \mathrm{cl}_\tau \widetilde{V}$.

Since $F = AH(F) = H(F)A$, we have

$$\begin{aligned}
\overline{\mathcal{T}(\{x'\} \times U)} &\supseteq u \circ (\{x'\} \times U) \supseteq u \circ (\{x'\} \times Vx) \\
&\supseteq \{ux'\} \times u(u \circ V)x \supseteq \{ux'\} \times u(u \circ (V \cap F))x \\
&= \{ux'\} \times \mathrm{cl}_\tau(V \cap F)x \supseteq \{ux'\} \times \mathrm{cl}_\tau \widetilde{V}x \\
&\supseteq \{ux'\} \times \alpha H(F)x = \{ux'\} \times \alpha H(F)Ax \\
&= \{ux'\} \times \alpha Fx = \{ux'\} \times Fx.
\end{aligned}$$

Since π is RIC, we have $u \circ Fx = \pi^{-1}(y)$. Hence

$$\overline{\mathcal{T}(\{x'\} \times U)} \supseteq u \circ (\{ux'\} \times Fx) = \{ux'\} \times \pi^{-1}(y).$$

This ends the proof of the claim.

Now it is easy to see that $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. Let $(x_1, x_2) \in R_\pi^n$, where $x_1 \in R_\pi^{n-1}$. Since x' is a transitive point of R_π^{n-1}, there exists a $p \in S_{\mathcal{T}}$ such that $px' = x_1$. Then $x_2 \in \pi^{-1}(py) = p \circ \pi^{-1}(y)$. Thus

$$(x_1, x_2) \in \{px'\} \times p \circ \pi^{-1}(y) \subseteq \overline{\mathcal{T}(\{ux'\} \times \pi^{-1}(y))} \subseteq \overline{\mathcal{T}(\{x'\} \times U)}.$$

Thus we have $R_\pi^n = \overline{\mathcal{T}(\{x'\} \times U)}$. □

Theorem B.3. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems and $y \in Y$. Then for all $n \geq 1$, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of R_π^n with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$.*

Proof. It is obvious for the case when $n = 1$. Now assume it is true for $n-1$. Fix a transitive point $x' = (x_1, x_2, \ldots, x_{n-1}) \in R_\pi^{n-1}$ with $x_1, x_2, \ldots, x_{n-1} \in \pi^{-1}(y)$. Assume that $y \in uY$ for some minimal idempotent $u \in J(\mathbf{M})$.

For each $\epsilon > 0$, define

$$V_\epsilon = \{x \in \overline{u\pi^{-1}(y)} : \mathcal{T}(x', x) \text{ is } \epsilon\text{-dense in } \; R_\pi^n\}.$$

It is easy to verify that V_ϵ is open. Now we show that V_ϵ is dense in $\overline{u\pi^{-1}(y)}$. For any $\Lambda \subseteq X^n, z \in X^n, \delta > 0$, $\Lambda \overset{\delta}{\sim} z$ is defined by $d(z, z') < \delta, \forall z' \in \Lambda$.

Now let $\{z_1, z_2, \cdots, z_n\}$ be an ϵ-net of R_π^n, i.e. for each $z \in R_\pi^n$ there is some z_j ($j \in \{1, 2, \ldots, n\}$) such that $d(z, z_j) < \epsilon$. Let U be an open subset of $\overline{w\pi^{-1}(y)}$. By Lemma B.2, $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. So there are some open subset $U_1 \supseteq U$ and $t_1 \in \mathcal{T}$ such that $t_1(\{x'\} \times U_1) \overset{\epsilon}{\sim} z_1$. Again, by Lemma B.2, $\overline{\mathcal{T}(\{x'\} \times U_1)} = R_\pi^n$. So there are an open subset $U_2 \supseteq U_1$ and $t_2 \in \mathcal{T}$ such that $t_2(\{x'\} \times U_2) \overset{\epsilon}{\sim} z_2$. ... Inductively, we have a sequence $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n$ (relatively open) and $t_1, \ldots, t_n \in \mathcal{T}$ such that $t_j(\{x'\} \times U_n) \overset{\epsilon}{\sim} z_j, \forall j \in \{1, 2, \ldots, n\}$. Hence $U_n \subseteq V_\epsilon$. This means that V_ϵ is dense in $\overline{u\pi^{-1}(y)}$.

Let $\Gamma = \bigcap_{n=1}^\infty V_{1/n}$. Then Γ is a residual set of $\overline{u\pi^{-1}(y)}$, and for all $x \in \Gamma$, we have $\overline{\mathcal{T}(x', x)} = R_\pi^n$. In particular, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of R_π^n with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$. The proof is completed. □

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REGIONALLY PROXIMAL RELATION OF ORDER d IS AN EQUIVALENCE ONE FOR MINIMAL SYSTEMS AND A COMBINATORIAL CONSEQUENCE

SONG SHAO AND XIANGDONG YE

ABSTRACT. By proving the minimality of face transformations acting on the diagonal points and searching the points allowed in the minimal sets, it is shown that the regionally proximal relation of order d, $\mathbf{RP}^{[d]}$, is an equivalence relation for minimal systems. Moreover, the lifting of $\mathbf{RP}^{[d]}$ between two minimal systems is obtained, which implies that the factor induced by $\mathbf{RP}^{[d]}$ is the maximal d-step nilfactor. The above results extend the same conclusions proved by Host, Kra and Maass for minimal distal systems.

A combinatorial consequence is that if S is a dynamically syndetic subset of $\mathbb{Z}$, then for each $d \geq 1$,

$$\{(n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic. In some sense this is the topological correspondence of the result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.

1. INTRODUCTION

The background of our study can be seen both in ergodic theory and topological dynamics.

1.1. **Background in ergodic theory.** The connection between ergodic theory and additive combinatorics was built in the 1970's with Furstenberg's beautiful proof of Szemerédi's theorem via ergodic theory [10]. Furstenberg's proof paved the way for obtaining new combinatorial results using ergodic methods, as well as leading to numerous developments within ergodic theory. Roughly speaking, Furstenberg [10] proved Szemerédi's theorem via the following ergodic theorem: let T be a measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$, and let $A \in \mathcal{B}$ with positive measure. Then for every integer $d \geq 1$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-dn}A) > 0.$$

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So it is natural to ask about the convergence of these averages, or more generally about the convergence in $L^2(X,\mu)$ of the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x),$$

where $f_1,\dots,f_d \in L^\infty(X,\mu)$. After nearly 30 years' efforts of many researchers, this problem was finally solved in [19, 30].

In their proofs the notion of characteristic factors plays a great role. Let us see why this notion is important. Loosely speaking, the structure theorem of [19, 30] states that if one wants to understand the multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)\dots f_d(T^{dn}x),$$

one can replace each function f_i by its conditional expectation on some d-step nilsystem (1-step nilsystem is the Kroneker's one). Thus one can reduce the problem to the study of the same average in a nilsystem, i.e. reducing the average in an arbitrary system to a more tractable question. Note that the multiple ergodic average for commuting transformations was obtained by Tao [26] using finitary ergodic method, see [3, 18] for more traditional ergodic proofs. Unfortunately, in this more general setting, the characteristic factors are not known up till now.

In [19], some useful tools, such as dynamical parallelepipeds, ergodic uniformity seminorms etc., were introduced in the study of dynamical systems. Their further applications were discussed in [18, 20, 21, 22, 23]. Now a natural and important question is what the topological correspondence of characteristic factors is. The history how to obtain the topological counterpart of characteristic factors will be discussed in the next subsection.

1.2. **Background in topological dynamics.** In some sense an equicontinuous system is the simplest system in topological dynamics. In the study of topological dynamics, one of the first problems was to characterize the equicontinuous structure relation $S_{eq}(X)$ of a system (X,T); i.e. to find the smallest closed invariant equivalence relation $R(X)$ on (X,T) such that $(X/R(X),T)$ is equicontinuous. A natural candidate for $R(X)$ is the so-called regionally proximal relation $\mathbf{RP}(X)$ [6]. By the definition, $\mathbf{RP}(X)$ is closed, invariant, and reflexive, but not necessarily transitive. The problem was then to find conditions under which $\mathbf{RP}(X)$ is an equivalence relation. It turns out to be a difficult problem. Starting with Veech [27], various authors, including MacMahon [25], Ellis-Keynes [8], came up with various sufficient conditions for $\mathbf{RP}(X)$ to be an equivalence relation. For somewhat different approach, see [2]. Note that in our case, $T: X\to X$ being homeomorphism and (X,T) being minimal, $\mathbf{RP}(X)$ is always an equivalence relation.

In [23] Host and Maass tried to find the topological counterpart of characteristic factors and obtained their goal in some particular case. Recently, Host, Kra and Maass [22] continued the study and succeeded for all minimal distal systems, which can be viewed as an analog of the purely ergodic structure theory of [10, 19, 30]. Note

that previously the counterpart of characteristic factors in topological dynamics was studied by Glasner [14, 15], where he considered the characterization of nil-system of order 2 in [14] and studied the characteristic factors for the action $T\times T^2\times\ldots\times T^n$ in [15]. To get the characteristic factors in topological dynamics, in [22, 23], for distal minimal systems a certain generalization of the regionally proximal relation is used to produce the maximal nilfactors.

Here is the notion of the regionally proximal relation of order d defined in [23, 22].

Definition 1.1. Let (X,T) be a system and let $d\geq 1$ be an integer. A pair $(x,y)\in X\times X$ is said to be *regionally proximal of order* d if for any $\delta>0$, there exist $x',y'\in X$ and a vector $\mathbf{n}=(n_1,\ldots,n_d)\in\mathbb{Z}^d$ such that $\rho(x,x')<\delta,\rho(y,y')<\delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon}x',T^{\mathbf{n}\cdot\epsilon}y')<\delta \text{ for any } \epsilon\in\{0,1\}^d,\ \epsilon\neq(0,\ldots,0),$$

where $\mathbf{n}\cdot\epsilon=\sum_{i=1}^d\epsilon_i n_i$. The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order* d.

It is easy to see that $\mathbf{RP}^{[d]}(X)$ is a closed and invariant relation for all $d\in\mathbb{N}$. When $d=1$, $\mathbf{RP}^{[d]}(X)$ is nothing but the classical regionally proximal relation. In [22], for distal minimal systems the authors showed that $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation, and the quotient of X under this relation is its maximal d-step nilfactor. So it remains the question open: is $\mathbf{RP}^{[d]}(X)$ an equivalence relation for any minimal system? The purpose of the current paper is to settle down the question.

1.3. **Main results.** In this article, we show that for each minimal system $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation and the quotient of X under this relation is its maximal d-step nilfactor.

Note that a subset S of $\mathbb{Z}$ is *dynamically syndetic* if there is a minimal system (X,T), $x\in X$ and an open neighborhood U of x such that $S=\{n\in\mathbb{Z}:T^nx\in U\}$. Equivalently, $S\subset\mathbb{Z}$ is dynamically syndetic if and only if S contains $\{0\}$ and 1_S is a minimal point of $(\{0,1\}^{\mathbb{Z}},\sigma)$, where σ is the shift map. A subset S of $\mathbb{Z}^d$ is *syndetic* if there exists a finite subset $F\subset\mathbb{Z}^d$ such that $S+F=\mathbb{Z}^d$. A combinatorial consequence of our results is that if S is a dynamically syndetic subset of $\mathbb{Z}$, then for each $d\geq1$,

$$\{\mathbf{n}=(n_1,\ldots,n_d)\in\mathbb{Z}^d:\epsilon\cdot\mathbf{n}=n_1\epsilon_1+\cdots+n_d\epsilon_d\in S,\epsilon_i\in\{0,1\},1\leq i\leq d\}$$

is syndetic. In some sense this is the topological correspondence of the following result obtained by Host and Kra for positive upper Banach density subsets using ergodic methods.

Theorem 1.2. [19, Theorem 1.5] *Let $A\subset\mathbb{Z}$ with $\overline{d}(A)>\delta>0$ and let $d\in\mathbb{N}$, then*

$$\{\mathbf{n}=(n_1,n_2,\ldots,n_k)\in\mathbb{Z}^d:\overline{d}\Big(\bigcap_{\epsilon\in\{0,1\}^d}(A+\epsilon\cdot\mathbf{n})\Big)\geq\delta^{2^d}\}$$

is syndetic, where $\overline{d}(B)$ denotes the upper density of $B\subset\mathbb{Z}$.

In [22] the authors showed that the regionally proximal relation of order d is an equivalence relation for minimal distal systems without using the enveloping semigroup theory except one known result that the distal extension between minimal systems is open (which is proved using the theory). In our situation we are forced to use the theory. The main idea of the proof is the following. First using the structure theory of a minimal system we show that the face transformations acting on the diagonal points are minimal, and then we prove some equivalence conditions for a pair being regionally proximal of order d. A key lemma here is to switch from a cubic point to a face point. Combining the minimality with the conditions we show that the regionally proximal relation of order d is an equivalence relation for minimal systems. Finally we show that $\mathbf{RP}^{[d]}$ can be lifted up from a factor to an extension between two minimal systems, which implies that the factor induced by $\mathbf{RP}^{[d]}$ is the maximal d-step nilfactor. It will be nice if one could have a proof of the minimality of face actions on the diagonal points without using the structure theory of minimal flows.

We remark that many results of the paper can be extended to abelian group actions.

1.4. **Organization of the paper.** In Section 2, we introduce some basic notions used in the paper. Since we will use tools from abstract topological dynamics, we collect basic facts about it in Appendix A. In Section 3, main results of the paper are discussed. The three sections followed are devoted to give proofs of main results. Note that lots of results obtained there have their independent interest. In the final section some applications are given.

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2. Preliminaries

In this section we introduce notions about dynamical parallelepipeds and nilsystems etc.. For more details see [19, 22].

2.1. **Topological dynamical systems.** A *transformation* of a compact metric space X is a homeomorphism of X to itself. A *topological dynamical system*, referred to more succinctly as just a *system*, is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a transformation. We use $\rho(\cdot, \cdot)$ to denote the metric in X. We also make use of a more general definition of a topological system. That is, instead of just a single transformation T, we will consider a countable abelian group of transformations. We collect basic facts about topological dynamics under general group actions in Appendix A.

A system (X, T) is *transitive* if there exists some point $x \in X$ whose orbit $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X and we call such a point a *transitive point*. The system is *minimal* if the orbit of any point is dense in X. This property

is equivalent to say that X and the empty set are the only closed invariant sets in X.

2.2. **Cubes and faces.** Let X be a set, let $d \geq 1$ be an integer, and write $[d] = \{1, 2, \ldots, d\}$. We view $\{0,1\}^d$ in one of two ways, either as a sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ of $0'$s and $1'$s; or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\mathbf{0} = (0, 0, \ldots, 0) \in \{0,1\}^d$ is the same to $\emptyset \subset [d]$.

If $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0,1\}^d$, we define

$$\mathbf{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i.$$

If we consider ϵ as $\epsilon \subset [d]$, then $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

We denote X^{2^d} by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon : \epsilon \in \{0,1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_\emptyset = x_{\mathbf{0}}$ is the first coordinate of $\mathbf{x}$. As examples, points in $X^{[2]}$ are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}),$$

and points in $X^{[3]}$ are like

$$\begin{aligned} & (x_{000}, x_{100}, x_{010}, x_{110}, x_{001}, x_{101}, x_{011}, x_{111}) \\ = \; & (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}, x_{\{3\}}, x_{\{1,3\}}, x_{\{2,3\}}, x_{\{1,2,3\}}). \end{aligned}$$

For $x \in X$, we write $x^{[d]} = (x, x, \ldots, x) \in X^{[d]}$. The diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0,1\}^{d-1})$ and $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0,1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d - 1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\epsilon : \epsilon \neq \emptyset) \in X_*^{[d]}$.

Identifying $\{0,1\}^d$ with the set of vertices of the Euclidean unit cube, a Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $x \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$. For details see [19].

2.3. **Dynamical parallelepipeds.**

Definition 2.1. Let (X, T) be a topological dynamical system and let $d \geq 1$ be an integer. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\mathbf{n} \cdot \epsilon} x = T^{n_1 \epsilon_1 + \ldots + n_d \epsilon_d} x : \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d),$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. When there is no ambiguity, we write $\mathbf{Q}^{[d]}$ instead of $\mathbf{Q}^{[d]}(X)$. An element of $\mathbf{Q}^{[d]}(X)$ is called a (dynamical) *parallelepiped of dimension d*.

It is important to note that $\mathbf{Q}^{[d]}$ is invariant under the Euclidean permutations of $X^{[d]}$.

As examples, $\mathbf{Q}^{[2]}$ is the closure in $X^{[2]} = X^4$ of the set

$$\{(x, T^m x, T^n x, T^{n+m} x) : x \in X, m, n \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}$ is the closure in $X^{[3]} = X^8$ of the set

$$\{(x, T^m x, T^n x, T^{m+n} x, T^p x, T^{m+p} x, T^{n+p} x, T^{m+n+p} x) : x \in X, m, n, p \in \mathbb{Z}\}.$$

Definition 2.2. Let $\phi : X \to Y$ and $d \in \mathbb{N}$. Define $\phi^{[d]} : X^{[d]} \to Y^{[d]}$ by $(\phi^{[d]}\mathbf{x})_\epsilon = \phi x_\epsilon$ for every $\mathbf{x} \in X^{[d]}$ and every $\epsilon \subset [d]$.

Let (X, T) be a system and $d \geq 1$ be an integer. The *diagonal transformation* of $X^{[d]}$ is the map $T^{[d]}$.

Definition 2.3. *Face transformations* are defined inductively as follows: Let $T^{[0]} = T$, $T^{[1]}_1 = \mathrm{id} \times T$. If $\{T^{[d-1]}_j\}_{j=1}^{d-1}$ is defined already, then set

$$T^{[d]}_j = T^{[d-1]}_j \times T^{[d-1]}_j, \ j \in \{1, 2, \ldots, d-1\},$$
$$T^{[d]}_d = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $j \in [d]$, the face transformation $T^{[d]}_j : X^{[d]} \to X^{[d]}$ can be defined by, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \subset [d]$,

$$T^{[d]}_j \mathbf{x} = \begin{cases} (T^{[d]}_j \mathbf{x})_\epsilon = T x_\epsilon, & j \in \epsilon; \\ (T^{[d]}_j \mathbf{x})_\epsilon = x_\epsilon, & j \notin \epsilon. \end{cases}$$

The *face group* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For $\mathcal{G}^{[d]}$ and $\mathcal{F}^{[d]}$, we use similar notations to that used for $X^{[d]}$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0,1\}^d)$. In particular, $\mathcal{F}^{[d]} = \{S \in \mathcal{G}^{[d]} : S_\emptyset = \mathrm{id}\}$.

For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$.

It is easy to verify that $\mathbf{Q}^{[d]}$ is the closure in $X^{[d]}$ of

$$\{S x^{[d]} : S \in \mathcal{F}^{[d]}, x \in X\}.$$

If x is a transitive point of X, then $\mathbf{Q}^{[d]}$ is the closed orbit of $x^{[d]}$ under the group $\mathcal{G}^{[d]}$.

2.4. **Nilmanifolds and nilsystems.** Let G be a group. For $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h and we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $k \geq 1$ be an integer. We say that G is *k-step nilpotent* if G_{k+1} is the trivial subgroup.

Let G be a k-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a *k-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$ of X. Then (X, T, μ) is called a *basic k-step nilsystem*. When the measure is not needed for results, we omit and write that (X, T) is a basic k-step nilsystem.

We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $(X_i, T_i)_{i\in\mathbb{N}}$ are systems with $diam(X_i) \le M < \infty$ and $\phi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\underleftarrow{\lim}\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x, y) = \sum_{i\in\mathbb{N}} 1/2^i d_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

Theorem 2.4 (Host-Kra-Maass)**.** [22, Theorem 1.2] *Assume that (X, T) is a transitive topological dynamical system and let $d \ge 2$ be an integer. The following properties are equivalent:*

(1) *If $x, y \in \mathbf{Q}^{[d]}(X)$ have $2^d - 1$ coordinates in common, then $x = y$.*
(2) *If $x, y \in X$ are such that $(x, y, \dots, y) \in \mathbf{Q}^{[d]}(X)$, then $x = y$.*
(3) *X is an inverse limit of basic $(d-1)$-step minimal nilsystems.*

A transitive system satisfying either of the equivalent properties above is called a $(d-1)$-step nilsystem *or a* system of order $(d-1)$.

2.5. Definition of the regionally proximal relations.

Definition 2.5. Let (X, T) be a system and let $d \ge 1$ be an integer. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta, \rho(y, y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon} x', T^{\mathbf{n}\cdot\epsilon} y') < \delta \text{ for any nonempty } \epsilon \subset [d].$$

(In other words, there exists $S \in \mathcal{F}^{[d]}$ such that $\rho(S_\epsilon x', S_\epsilon y') < \delta$ for every $\epsilon \neq \emptyset$.) The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}$ (or by $\mathbf{RP}^{[d]}(X)$ in case of ambiguity), which is called *the regionally proximal relation of order d*.

It is easy to see that $\mathbf{RP}^{[d]}$ is a closed and invariant relation for all $d \in \mathbb{N}$. Note that

$$\ldots \subseteq \mathbf{RP}^{[d+1]} \subseteq \mathbf{RP}^{[d]} \subseteq \ldots \subseteq \mathbf{RP}^{[2]} \subseteq \mathbf{RP}^{[1]} = \mathbf{RP}(X).$$

By the definition it is easy to verify the following equivalent condition for $\mathbf{RP}^{[d]}$, see [22].

Lemma 2.6. *Let (X, T) be a minimal system and let $d \ge 1$ be an integer. Let $x, y \in X$. Then $(x, y) \in \mathbf{RP}^{[d]}$ if and only if there is some $\mathbf{a}_* \in X_*^{[d]}$ such that $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$.*

Remark 2.7. When $d = 1$, $\mathbf{RP}^{[1]}$ is the classical regionally proximal relation. If (X, T) is minimal, it is easy to verify directly the following useful fact:

$$(x, y) \in \mathbf{RP} = \mathbf{RP}^{[1]} \Leftrightarrow (x, x, y, x) \in \mathbf{Q}^{[2]} \Leftrightarrow (x, y, y, y) \in \mathbf{Q}^{[2]}.$$

3. Main results

In this section we will state the main results of the paper.

3.1. $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$.

To show $\mathbf{RP}^{[d]}$ is an equivalence relation we are forced to investigate the $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and the equivalent conditions for $\mathbf{RP}^{[d]}$. Those are done in Theorem 3.1 and Theorem 3.2 respectively.

First recall that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system, which is mentioned in [22]. But we need to know $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$. Let (X, T) be a system and $x \in X$. Recall that $\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$ for $\mathbf{x} \in X^{[d]}$. Set

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

Theorem 3.1. *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then*

(1) $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ *is minimal for all* $x \in X$.
(2) $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ *is the unique* $\mathcal{F}^{[d]}$*-minimal subset in* $\mathbf{Q}^{[d]}[x]$ *for all* $x \in X$.

3.2. $\mathbf{RP}^{[d]}$ is an equivalence relation.

With the help of Theorem 3.1, we can prove that $\mathbf{RP}^{[d]}$ is an equivalence relation. First we have the following equivalent conditions for $\mathbf{RP}^{[d]}$.

Theorem 3.2. *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) $(x, y) \in \mathbf{RP}^{[d]}$;
(2) $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$;
(3) $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

Proof. (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1) follows from Lemma 2.6. Hence it suffices to show (1) $\Rightarrow$ (3).

Let $(x, y) \in \mathbf{RP}^{[d]}$. Then by Lemma 2.6 there is some $\mathbf{a}_* \in X_*^{[d]}$ such that $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$. Observe that $(y, \mathbf{a}_*) \in \mathbf{Q}^{[d]}$. By Theorem 3.1-(2), there is a sequence $\{F_k\} \subset \mathcal{F}^{[d]}$ such that $F_k(y, \mathbf{a}_*) \to y^{[d]}, k \to \infty$. Hence

$$F_k \times F_k(x, \mathbf{a}_*, y, \mathbf{a}_*) \to (x, y_*^{[d]}, y, y_*^{[d]}) = (x, y_*^{[d+1]}),\ k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $(x, \mathbf{a}_*, y, \mathbf{a}_*) \in \mathbf{Q}^{[d+1]}$, we have that $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}$.

By Theorem 3.1-(1), $y^{[d+1]}$ is $\mathcal{F}^{[d+1]}$-minimal. It follows that $(x, y_*^{[d+1]})$ is also $\mathcal{F}^{[d+1]}$-minimal. Now $(x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}[x]$ is $\mathcal{F}^{[d+1]}$-minimal and by Theorem 3.1-(2), $\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ is the unique $\mathcal{F}^{[d+1]}$-minimal subset in $\mathbf{Q}^{[d+1]}[x]$. Hence we have that $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$, and the proof is completed. □

By Theorem 3.2, we have the following theorem immediately.

Theorem 3.3. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]}(X)$ is an equivalence relation.*

Proof. It suffices to show the transitivity, i.e. if $(x,y),(y,z) \in \mathbf{RP}^{[d]}(X)$, then $(x,z) \in \mathbf{RP}^{[d]}(X)$. Since $(x,y),(y,z) \in \mathbf{RP}^{[d]}(X)$, by Theorem 3.2 we have

$$(y,x,x,\dots,x),(y,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}).$$

By Theorem 3.1 $(\overline{\mathcal{F}^{[d+1]}}(y^{[d+1]}),\mathcal{F}^{[d+1]})$ is minimal, it follows that $(y,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(y,x,x,\dots,x)$. It follows that $(x,z,z,\dots,z) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. By Theorem 3.2 again, $(x,z) \in \mathbf{RP}^{[d]}(X)$. □

Remark 3.4. By Theorem 3.2 we know that in the definition of regionally proximal relation of d, x' can be replaced by x. More precisely, $(x,y) \in \mathbf{RP}^{[d]}$ if and only if for any $\delta > 0$ there exist $y' \in X$ and a vector $\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d$ such that for any nonempty $\epsilon \subset [d]$

$$\rho(y,y') < \delta \text{ and } \rho(T^{\mathbf{n}\cdot\epsilon}x, T^{\mathbf{n}\cdot\epsilon}y') < \delta.$$

3.3. **$\mathbf{RP}^{[d]}$ and nilfactors.** A subset $S \subset \mathbb{Z}$ is *thick* if it contains arbitrarily long runs of positive integers, i.e. there is a subsequence $\{n_i\}$ of $\mathbb{Z}$ such that $S \supset \bigcup_{i=1}^{\infty}\{n_i, n_i+1,\dots,n_i+i\}$.

Let $\{b_i\}_{i\in I}$ be a finite or infinite sequence in $\mathbb{Z}$. One defines

$$FS(\{b_i\}_{i\in I}) = \Big\{ \sum_{i\in\alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \Big\}$$

Note when $I = [d]$,

$$FS(\{b_i\}_{i=1}^{d}) = \Big\{ \sum_{i\in I} b_i\epsilon_i : \epsilon = (\epsilon_i) \in \{0,1\}^d \setminus \{\emptyset\} \Big\}.$$

F is an *IP set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, where $p_i \in \mathbb{Z}$.

Lemma 3.5. *Let (X,T) be a system. Then for every $d \in \mathbb{N}$, the proximal relation*

$$\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X).$$

Proof. Let $(x,y) \in \mathbf{P}(X)$ and $\delta > 0$. Set

$$N_\delta(x,y) = \{n \in \mathbb{Z} : \rho(T^n x, T^n y) < \delta\}.$$

It is easy to check $N_\delta(x,y)$ is thick and hence an IP set. From this it follows that $\mathbf{P}(X) \subseteq \mathbf{RP}^{[d]}(X)$. More precisely, set $FS(\{p_i\}_{i=1}^{\infty}) \subseteq N_\delta(x,y)$, then for any $d \in \mathbb{N}$,

$$\rho(T^{p_1\epsilon_1+\dots+p_d\epsilon_d}x, T^{p_1\epsilon_1+\dots+p_d\epsilon_d}y) < \delta,\ \epsilon = (\epsilon_1,\dots,\epsilon_d) \in \{0,1\}^d, \epsilon \neq (0,\dots,0).$$

That is, $(x,y) \in \mathbf{RP}^{[d]}$ for all $d \in \mathbb{N}$. □

The following corollary was observed in [23] for $d = 2$.

Corollary 3.6. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then (X,T) is a weakly mixing system if and only if*

$$\mathbf{RP}^{[d]} = X \times X.$$

Proof. Since a minimal system (X,T) is weakly mixing if and only if $\overline{\mathbf{P}(X)} = \mathbf{RP}(X) = X \times X$ (see [1]), so the result follows from Lemma 3.5. □

We remark that more interesting properties for weakly mixing systems will be shown in Theorem 3.11 in the sequel.

Proposition 3.7. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then $\mathbf{RP}^{[d]} = \Delta$ if and only if X is a system of order d.*

Proof. It follows from Theorem 3.2 and Theorem 2.4 directly. □

3.4. **Maximal nilfactors.** Note that the lifting property of $\mathbf{RP}^{[d]}$ between two minimal systems is obtained in the paper. This result is new even for minimal distal systems.

Theorem 3.8. *Let $\pi : (X,T) \to (Y,T)$ be a factor map and $d \in \mathbb{N}$. Then*

(1) $\pi \times \pi(\mathbf{RP}^{[d]}(X)) \subseteq \mathbf{RP}^{[d]}(Y)$;

(2) *if (X,T) is minimal, then $\pi \times \pi(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.*

Proof. (1) It follows from the definition.

(2) It will be proved in Section 6. □

Theorem 3.9. *Let $\pi : (X,T) \to (Y,T)$ be a factor map of minimal systems and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *(Y,T) is a d-step nilsystem;*

(2) $\mathbf{RP}^{[d]}(X) \subset R_\pi$.

Especially the quotient of X under $\mathbf{RP}^{[d]}(X)$ is the maximal d-step nilfactor of X, i.e. any d-step nilfactor of X is the factor of $X/\mathbf{RP}^{[d]}(X)$.

Proof. Assume that (Y,T) is a d-step nilsystem. Then we have $\mathbf{RP}^{[d]}(Y) = \Delta_Y$ by Proposition 3.7. Hence by Theorem 3.8-(1),

$$\mathbf{RP}^{[d]}(X) \subset (\pi \times \pi)^{-1}(\Delta_Y) = R_\pi.$$

Conversely, assume that $\mathbf{RP}^{[d]}(X) \subset R_\pi$. If (Y,T) is not a d-step nilsystem, then by Proposition 3.7, $\mathbf{RP}^{[d]}(Y) \neq \Delta_Y$. Let $(y_1,y_2) \in \mathbf{RP}^{[d]} \setminus \Delta_Y$. Now by Theorem 3.8, there are $x_1, x_2 \in X$ such that $(x_1,x_2) \in \mathbf{RP}^{[d]}(X)$ with $(\pi \times \pi)(x_1,x_2) = (y_1,y_2)$. Since $\pi(x_1) = y_1 \neq y_2 = \pi(x_2)$, $(x_1,x_2) \notin R_\pi$. This means that $\mathbf{RP}^{[d]}(X) \not\subset R_\pi$, a contradiction! The proof is completed. □

Remark 3.10. In [22, Proposition 4.5] it is showed that this proposition holds for minimal distal systems.

3.5. **Weakly mixing systems.** In this subsection we completely determine $\mathbf{Q}^{[d]}$ and $\overline{\mathcal{F}^{[d]}}(x^{[d]})$ for minimal weakly mixing systems.

Theorem 3.11. *Let (X,T) be a minimal weakly mixing system and $d \geq 1$. Then*

(1) *$(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$;*

(2) *For all $x \in X$, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal and*

$$\overline{\mathcal{F}^{[d]}}(x^{[d]}) = \{x\} \times X_*^{[d]} = \{x\} \times X^{2^d-1}.$$

Proof. The fact that $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is minimal and $\mathbf{Q}^{[d]} = X^{[d]}$ is followed from (2) easily. Hence it suffices to show (2).

We will show for any point of $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$, we have

$$\overline{\mathcal{F}^{[d]}}(\mathbf{x}) = \{x\} \times X_*^{[d]},$$

which obviously implies (2). First note that it is trivial for $d = 1$. Now we assume that (1), and hence (2) hold for $d-1$, $d \geq 2$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbf{Q}^{[d]}$. Since (X, T) is weakly mixing, $(X^{[d-1]}, T^{[d-1]})$ is transitive (see [9]). Let $\mathbf{a} \in X^{[d-1]}$ be a transitive point. By the induction for $d-1$, $\mathbf{Q}^{[d-1]} = X^{[d-1]}$ is $\mathcal{G}^{[d]}$-minimal. Hence $\mathbf{a} \in \overline{\mathcal{O}(\mathbf{x}'', \mathcal{G}^{[d-1]})}$ and there is some sequence $F_k \in \mathcal{F}^{[d]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k\mathbf{x} = F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}),\ k \to \infty.$$

Especially $(\mathbf{w}, \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Note that

$$(T_d^{[d]})^n(\mathbf{w}, \mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n\mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x})$$

Since $\mathbf{a}$ is a transitive point of $(X^{[d-1]}, T^{[d-1]})$, we have

$$\{\mathbf{w}\} \times X^{[d-1]} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.1}$$

By the induction assumption for $d-1$, $\mathbf{w}$ is minimal for $\mathcal{F}^{[d-1]}$ action and

$$\overline{\mathcal{F}^{[d-1]}}(\mathbf{w}) = \overline{\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]})} = \{x\} \times X_*^{[d-1]}. \tag{3.2}$$

By acting the elements of $\mathcal{F}^{[d]}$ on (3.1), we have

$$\mathcal{O}(\mathbf{w}, \mathcal{F}^{[d-1]}) \times X^{[d-1]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\{x\} \times X_*^{[d-1]} \times X^{[d-1]} = \{x\} \times X_*^{[d]} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

This completes the proof. □

Remark 3.12. We remark that using the so-called natural extension, it can be shown that the main results of the paper hold for continuous surjective maps.

4. $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$

In this section we discuss $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}$ and prove Theorem 3.1-(1). First we will discuss proximal extensions, distal extensions and weakly mixing extension one by one. They exhibit different properties and satisfy our requests by different reasons. After that, the proof of Theorem 3.1-(1) will be given. The proof of Theorem 3.1-(2) will be given in next section. For notions which are not mentioned before see Appendix A.

4.1. **Idea of the proof of Theorem 3.1-(1).** Before going on let us say something about the idea in the proof of Theorem 3.1-(1). By the structure theorem A.6, for a minimal system (X, T), we have the following diagram.

$$\begin{array}{ccc} X_\infty & \xrightarrow{\pi} & X \\ \big\downarrow{\scriptstyle \phi} & & \\ Y_\infty & & \end{array}$$

In this diagram Y_∞ is a strictly PI system, ϕ is weakly mixing and RIC, and π is proximal.

So if we want to show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$, it is sufficient to show it holds for X_∞. By the definition of X_∞ and Y_∞, it is sufficient to consider the following cases: (1) proximal extensions; (2) distal or equicontinuous extensions; (3) RIC weakly mixing extensions and (4) the inverse limit. Since the inverse limit is easy to handle, we need only focus on the three kinds of extensions.

4.2. **Properties about three kinds of extensions.** In this subsection we collect some properties about proximal, distal and weakly mixing extensions, which will be used frequently in the sequel. As in Appendix A, $(X, \mathcal{T})$ is a system under the action of a topological group $\mathcal{T}$, and $E(X, \mathcal{T})$ is its enveloping semigroup.

The following two lemmas are folk results, for completeness we include proofs.

Lemma 4.1. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a proximal extension of minimal systems. Let $x \in X, y = \pi(x)$ and let $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$. Then there is some $p \in E(X, \mathcal{T})$ such that*

$$px_1 = px_2 = \ldots = px_n = x.$$

Especially, when $x = x_1$, we have that $(x_1, x_2, \ldots, x_n)$ is proximal to $(x, x, \ldots, x)$ in $(X^n, \mathcal{T})$.

Proof. Since $(x_1, x_2) \in R_\pi \subset \mathbf{P}(X, \mathcal{T})$, by Proposition A.3 there is some $p \in E(X, \mathcal{T})$ such that $px_1 = px_2$.

Now assume that for $2 \le j \le n-1$, there is some $p_1 \in E(X, \mathcal{T})$ such that $p_1x_1 = p_1x_2 = \ldots = p_1x_j$. Since R_π is closed and invariant and $(x_j, x_{j+1}) \in R_\pi$, $(p_1x_j, p_1x_{j+1}) \in R_\pi \subset \mathbf{P}(X, \mathcal{T})$. So by Proposition A.3 there is $p_2 \in E(X, \mathcal{T})$ such that $p_2(p_1x_j) = p_2(p_1x_{j+1})$. Let $p = p_2p_1$, then we have

$$px_1 = px_2 = \ldots = px_j = px_{j+1}.$$

Inductively, there is some $p \in E(X, \mathcal{T})$ such that

$$px_1 = px_2 = \ldots = px_n.$$

Since $(X, \mathcal{T})$ is minimal, we can assume that they are equal to x.

If $x_1 = x$, then $px_1 = px_2 = \ldots = px_n = x = x_1$ and hence

$$p(x_1, x_2, \ldots, x_n) = (x, x, \ldots, x) = p(x, x, \ldots, x).$$

That is, $(x_1, x_2, \ldots, x_n)$ is proximal to $(x, x, \ldots, x)$ in $(X^n, \mathcal{T})$. $\square$

Lemma 4.2. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a distal extension of systems. Then for any $x \in X$, if $\pi(x)$ is minimal in $(Y, \mathcal{T})$, then x is minimal in $(X, \mathcal{T})$. Especially, if $(Y, \mathcal{T})$ is semi-simple (i.e. every point is minimal), then so is $(X, \mathcal{T})$.*

Proof. Let $x \in X$ and $y = \pi(x)$. Since y is a minimal point, by Proposition A.2 there is some minimal idempotent $u \in E(X, \mathcal{T})$ such that $uy = y$. Then $\pi(ux) = u\pi(x) = uy = y$. Hence $ux, x \in \pi^{-1}(y)$. Since $(ux, x) \in \mathbf{P}(X, \mathcal{T})$ (Proposition A.3) and π is distal, we have $ux = x$. That is, x is a minimal point of X by Proposition A.2. $\square$

Now we discuss weakly mixing extensions. We need Theorem 4.3, which is a generalization of [1, Chapter 14, Theorem 28]. Note that in [17, Theorem 2.7 and Corollary 2.9] Glasner showed that R_π^n is transitive. So Theorem 4.3 is a slight strengthening of the results in [17]. Since its proof needs some techniques in the enveloping semigroup theory, we leave it to the appendix.

Theorem 4.3. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems, then for all $n \geq 1$ and $y \in Y$, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of R_π^n with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$.*

Note that each RIC extension is open, and $\pi : X \to Y$ is open if and only if $Y \to 2^X, y \mapsto \pi^{-1}(y)$ is continuous, see for instance [29]. Using Theorem 4.3 we have the following lemma, which will be used in the sequel.

Lemma 4.4. *Let $\pi : (X, T) \to (Y, T)$ be a RIC weakly mixing extension of minimal systems of (X, T) and (Y, T). Then for each $y \in Y$ and $d \geq 1$, we have*

(1) $(\pi^{-1}(y))^{[d]} = (\pi^{-1}(y))^{2^d} \subset \mathbf{Q}^{[d]}(X)$,

(2) *for all $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$ and $\pi^{[d]}(\mathbf{x}) = y^{[d]}$*

$$\{x\} \times \left(\pi^{-1}(y)\right)_*^{[d]} = \{x\} \times \left(\pi^{-1}(y)\right)^{2^d - 1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Proof. The idea of proof is similar to Theorem 3.11. When $d = 1$, for any $(x, x') \in X^{[1]} = X \times X$, $\overline{\mathcal{F}^{[1]}}(x, x') = \overline{\mathcal{O}}\left((x, x'), \mathrm{id} \times T\right) = \{x\} \times X$ and $\mathbf{Q}^{[1]}(X) = X \times X$. Hence the results hold obviously. Now we show the case for $d = 2$. Let $\mathbf{x} = (x_1, x_2, x_3, x_4) \in X^{[2]}$ with $\pi^{[2]}(x_1, x_2, x_3, x_4) = y^{[2]}$. By Theorem 4.3, there is a transitive point (a, b) of $(R_\pi, T \times T)$ with $\pi(a) = \pi(b) = y$. Since (X, T) is minimal, there is some sequence $\{n_i\} \subset \mathbb{Z}$ such that $T^{n_i} x_3 \to a, i \to \infty$. Without loss of generality, assume that $T^{n_i} x_4 \to x_4', i \to \infty$ for some $x_4' \in X$. Since $\pi(a) = y$, $\pi(x_4') = y$ too. So

$$(\mathrm{id} \times \mathrm{id} \times T \times T)^{n_i}(x_1, x_2, x_3, x_4) \to (x_1, x_2, a, x_4'),\ i \to \infty. \tag{4.1}$$

Since (X, T) is minimal, there is some sequence $\{m_i\} \subset \mathbb{Z}$ such that $T^{m_i} x_4' \to b, i \to \infty$. Without loss of generality, assume that $T^{m_i} x_2 \to x_2', i \to \infty$ for some $x_2' \in X$. Since $\pi(b) = y$, $\pi(x_2') = y$ too. So

$$(\mathrm{id} \times T \times \mathrm{id} \times T)^{m_i}(x_1, x_2, a, x_4') \to (x_1, x_2', a, b),\ i \to \infty. \tag{4.2}$$

Hence by (4.1) and (4.2),

$$(x_1, x_2', a, b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.3}$$

Thus for all $n \in \mathbb{Z}$,

$$(x_1, x_2', T^n a, T^n b) = (\mathrm{id} \times \mathrm{id} \times T \times T)^n (x_1, x_2', a, b) \in \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since (a, b) is a transitive point of $(R_\pi, T \times T)$, it follows that

$$\{x_1\} \times \{x_2'\} \times \pi^{-1}(y) \times \pi^{-1}(y) \subset \{x_1\} \times \{x_2'\} \times R_\pi \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.4}$$

Now we show that

$$\{x_1\} \times \pi^{-1}(y) \times \pi^{-1}(y) \times \pi^{-1}(y) = \{x_1\} \times (\pi^{-1}(y))^3 \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}). \tag{4.5}$$

For any $z \in \pi^{-1}(y)$, there is a sequence $k_i \subset \mathbb{Z}$ such that $T^{k_i} x_2' \to z, i \to \infty$. Thus $T^{k_i} y = T^{k_i} \pi(x_2') = \pi(T^{k_i} x_2') \to \pi(z) = y, i \to \infty$. Since π is open, we have $T^{k_i} \pi^{-1}(y) = \pi^{-1}(T^{k_i} y) \to \pi^{-1}(y), i \to \infty$ in the Hausdorff metric. Thus

$$\{x_1\} \times \{z\} \times \pi^{-1}(y)^2 \subset \overline{\cup_{i=1}^{\infty} (\mathrm{id} \times T \times \mathrm{id} \times T)^{k_i} (\{x_1\} \times \{x_2'\} \times \pi^{-1}(y)^2)} \subset \overline{\mathcal{F}^{[2]}}(\mathbf{x}).$$

Since z is arbitrary, we have (4.5). Similarly, we have $(\pi^{-1}(y))^4 \subset \mathbf{Q}^{[2]}(X)$ and we are done for $d = 2$.

Now assume we have (1) and (2) for $d-1$ already, and show the case for d. Let $\mathbf{x} \in X^{[d]}$ with $x_\emptyset = x$ and $\pi^{[d]}(\mathbf{x}) = y^{[d]}$.

Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$. Since π is weakly mixing, $(R_\pi^{2^{d-1}}, T^{[d-1]})$ is transitive. By Theorem 4.3 there is $\mathbf{a} \in R_\pi^{2^{d-1}}$ which is a transitive point of $(R_\pi^{2^{d-1}}, T^{[d-1]})$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$. Without loss of generality, we may assume that $a_\emptyset = x_\emptyset''$ (i.e. the first coordinate of $\mathbf{a}$ is equal to that of $\mathbf{x}''$), otherwise we may use the face transformation $\mathrm{id}^{[d-1]} \times T^{[d-1]}$ to find some point in $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$ satisfying this property.

By the induction assumption for $d-1$,

$$\mathbf{a} \in \{x_\emptyset''\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{x}'').$$

Hence there is some sequence $F_k \in \mathcal{F}^{[d-1]}$ and $\mathbf{w} \in X^{[d-1]}$ such that

$$F_k \times F_k(\mathbf{x}) = F_k \times F_k(\mathbf{x}', \mathbf{x}'') \to (\mathbf{w}, \mathbf{a}),\ k \to \infty.$$

Especially $(\mathbf{w}, \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Since $\pi^{[d]}(\mathbf{x}) = y^{[d]}$ and $\pi^{[d-1]}(\mathbf{a}) = y^{[d-1]}$, it is easy to verify that $\pi^{[d-1]}(\mathbf{w}) = y^{[d-1]}$ and $w_\emptyset = x$. Note that

$$(T_d^{[d]})^n (\mathbf{w}, \mathbf{a}) = (\mathbf{w}, (T^{[d-1]})^n \mathbf{a}) \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

We have

$$\{\mathbf{w}\} \times \mathcal{O}(\mathbf{a}, T^{[d-1]}) \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

And so

$$\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} \subset \{\mathbf{w}\} \times R_\pi^{2^{d-1}} = \{\mathbf{w}\} \times \overline{\mathcal{O}(\mathbf{a}, T^{[d-1]})} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}). \tag{4.6}$$

By the induction assumption for $d-1$, for $\mathbf{w}$ we have

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \subset \overline{\mathcal{F}^{[d-1]}}(\mathbf{w}). \tag{4.7}$$

Hence for all $\mathbf{z} \in \{x\} \times (\pi^{-1}(y))^{2^{d-1}-1}$, there is some sequence $\{H_k\} \subset \mathcal{F}^{[d-1]}$ such that $H_k\mathbf{w} \to \mathbf{z}, k \to \infty$. Since π is open, similar to the proof of (4.5), we have that $H_k (\pi^{-1}(y))^{2^{d-1}} \to (\pi^{-1}(y))^{2^{d-1}}, k \to \infty$. Hence

$$H_k \times H_k \left(\{\mathbf{w}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}\right) \to \{\mathbf{z}\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}}, k \to \infty.$$

Since $H_k \times H_k \in \mathcal{F}^{[d]}$ and $\mathbf{z} \in \{x\} \times (\pi^{-1}(y))^{2^{d-1}-1}$ is arbitrary, it follows from (4.6) that

$$\{x\} \times \left(\pi^{-1}(y)\right)^{2^{d-1}-1} \times \left(\pi^{-1}(y)\right)^{2^{d-1}} = \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Now by this fact it is easy to get $(\pi^{-1}(y))^{[d]} = (\pi^{-1}(y))^{2^d} \subset \mathbf{Q}^{[d]}(X)$. So (1) and (2) hold for the case d. This completes the proof. □

In fact with a small modification of the above proof one can show that $R_\pi^{2^d} \subset \mathbf{Q}^{[d]}(X)$. We do not know if $\{x\} \times R_\pi^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{x})$.

4.3. **Proof of Theorem 3.1-(1).** A subset $S \subseteq \mathbb{Z}$ is a *central set* if there exists a system (X,T), a point $x \in X$ and a minimal point $y \in X$ proximal to x, and a neighborhood U_y of y such that $N(x,U_y) \subset S$, where $N(x,U_y) = \{n \in \mathbb{Z} : T^n x \in U_y\}$. It is known that any central set is an IP-set [11, Proposition 8.10.].

Proposition 4.5. *Let $\pi : (X,T) \to (Y,T)$ be a proximal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Especially there is some $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k\mathbf{y} \to y^{[d]},\ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k\mathbf{x} \to \mathbf{z},\ k \to \infty. \tag{4.8}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k\mathbf{x}) = \lim_k F_k\mathbf{y} = y^{[d]}$. That is,

$$z_\epsilon \in \pi^{-1}(y),\ \forall \epsilon \in \{0,1\}^d.$$

Since π is proximal, by Lemma 4.1 there is some $p \in E(X,T)$ such that

$$pz_\epsilon = px = x, \quad \forall \epsilon \in \{0,1\}^d.$$

That is, $p\mathbf{z} = x^{[d]} = px^{[d]}$, i.e. $\mathbf{z}$ is proximal to $x^{[d]}$ under the action of $T^{[d]}$. Since $x^{[d]}$ is $T^{[d]}$-minimal, for any neighborhood $\mathbf{U}$ of $x^{[d]}$,

$$N_{T^{[d]}}(\mathbf{z}, \mathbf{U}) = \{n \in \mathbb{Z} : (T^{[d]})^n\mathbf{z} \in \mathbf{U}\}$$

is a central set and hence contains some IP set $FS(\{p_i\}_{i=1}^{\infty})$. Particularly,

$$FS(\{p_i\}_{i=1}^{d}) \subseteq N_{T^{[d]}}(\mathbf{z}, \mathbf{U}).$$

This means for all $\epsilon \in \{0,1\}^d \setminus \{\mathbf{0}\}$,

$$(T^{[d]})^{\mathbf{p}\cdot\epsilon}\mathbf{z} \in \mathbf{U},$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}^d$. Especially,

$$(T^{\mathbf{p}\cdot\epsilon} z_\epsilon)_{\epsilon\in\{0,1\}^d} \in \mathbf{U}$$

In other words, we have

$$(T_1^{[d]})^{p_1}(T_2^{[d]})^{p_2}\ldots(T_d^{[d]})^{p_d}\mathbf{z} \in \mathbf{U}.$$

Since $\mathbf{U}$ is arbitrary, we have that $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{z})$. Combining with (4.8), we have

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Thus $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. This completes the proof. □

Proposition 4.6. *Let $\pi : (X,T) \to (Y,T)$ be a distal extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It follows from Lemma 4.2, since it is easy to check that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a distal extension. □

Proposition 4.7. *Let $\pi : (X,T) \to (Y,T)$ be a RIC weakly mixing extension of minimal systems and $d \in \mathbb{N}$. If $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $y \in Y$, then $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.*

Proof. It is sufficient to show that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. Let $y = \pi(x)$. Then by the assumption $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal. Note that $\pi^{[d]} : (\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]}) \to (\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is a factor map. Especially there is some $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ such that $\pi^{[d]}(\mathbf{x}) = \mathbf{y}$.

Since $\mathbf{y} \in \overline{\mathcal{F}^{[d]}}(y^{[d]})$ and $(\overline{\mathcal{F}^{[d]}}(y^{[d]}), \mathcal{F}^{[d]})$ is minimal, there is some sequence $F_k \in \mathcal{F}^{[d]}$ such that

$$F_k\mathbf{y} \to y^{[d]},\ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k\mathbf{x} \to \mathbf{z},\ k \to \infty. \tag{4.9}$$

Then $\pi^{[d]}(\mathbf{z}) = \lim_k \pi^{[d]}(F_k\mathbf{x}) = \lim_k F_k\mathbf{y} = y^{[d]}$. By Lemma 4.4

$$x^{[d]} \in \{x\} \times \left(\pi^{-1}(y)\right)^{2^d-1} \subset \overline{\mathcal{F}^{[d]}}(\mathbf{z}).$$

Together with (4.9), we have $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$. This completes the proof. □

Proof of Theorem 3.1-(1): By the structure theorem A.6, we have the following diagram, where Y_∞ is a strictly PI-system, ϕ is RIC weakly mixing extension and π is proximal.

$$\begin{array}{ccc} X_\infty & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle\phi} & & \\ Y_\infty & & \end{array}$$

Since the inverse limit of minimal systems is minimal, it follows from Propositions 4.5, 4.6 that the result holds for Y_∞. By Proposition 4.7 it also holds for X_∞. Since the factor of a minimal system is always minimal, it is easy to see that we have the theorem for X. $\square$

4.4. **Minimality of $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$.** We will need the following theorem mentioned in [22], where no proof is included. We give a proof (due to Glasner-Ellis) here for completeness. Note one can also prove this result using the method in the previous subsection.

Proposition 4.8. *Let (X,T) be a minimal system and let $d \geq 1$ be an integer. Let A be a $T^{[d]}$-minimal subset of $X^{[d]}$ and set $N = \overline{\mathcal{O}(A, \mathcal{F}^{[d]})} = \mathrm{cl}\big(\bigcup\{SA : S \in \mathcal{F}^{[d]}\}\big)$. Then $(N, \mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in N.*

Proof. The proof is similar to the one in [16]. Let $E = E(N, \mathcal{G}^{[d]})$ be the enveloping semigroup of $(N, \mathcal{G}^{[d]})$. Let $\pi_\epsilon : N \to X$ be the projection of N on the ϵ-th component, $\epsilon \in \{0,1\}^d$. We consider the action of the group $\mathcal{G}^{[d]}$ on the ϵ-th component via the representation $T^{[d]} \mapsto T$ and

$$T_j^{[d]} \mapsto \begin{cases} T, & j \in \epsilon; \\ \mathrm{id}, & j \notin \epsilon. \end{cases}$$

With respect to this action of $\mathcal{G}^{[d]}$ on X the map π_ϵ is a factor map $\pi_\epsilon : (N, \mathcal{G}^{[d]}) \to (X, \mathcal{G}^{[d]})$. Let $\pi_\epsilon^* : E(N, \mathcal{G}^{[d]}) \to E(X, \mathcal{G}^{[d]})$ be the corresponding homomorphism of enveloping semigroups. Notice that for this action of $\mathcal{G}^{[d]}$ on X clearly $E(X, \mathcal{G}^{[d]}) = E(X,T)$ as subsets of X^X.

Let now $u \in E(N, T^{[d]})$ be any minimal idempotent in the enveloping semigroup of $(N, T^{[d]})$. Choose v a minimal idempotent in the closed left ideal $E(N, \mathcal{G}^{[d]})u$. Then $vu = v$, i.e. $v <_L u$. Set for each $\epsilon \in \{0,1\}^d$, $u_\epsilon = \pi_\epsilon^* u$ and $v_\epsilon = \pi_\epsilon^* v$. We want to show that also $uv = u$, i.e. $u <_L v$. Note that as an element of $E(N, \mathcal{G}^{[d]})$ is determined by its projections, it suffices to show that for each $\epsilon \in \{0,1\}^d$, $u_\epsilon v_\epsilon = u_\epsilon$.

Since for each $\epsilon \in \{0,1\}^d$ the map π_ϵ^* is a semigroup homomorphism, we have $v_\epsilon u_\epsilon = v_\epsilon$ as $vu = v$. In particular we deduce that v_ϵ is an element of the minimal left ideal of $E(X,T)$ which contains u_ϵ. In turn this implies

$$u_\epsilon v_\epsilon = u_\epsilon v_\epsilon u_\epsilon = u_\epsilon;$$

and it follows that indeed $uv = u$. Thus u is an element of the minimal left ideal of $E(N, \mathcal{G}^{[d]})$ which contains v, and therefore u is a minimal idempotent of $E(N, \mathcal{G}^{[d]})$.

Now let x be an arbitrary point in A and let $u \in E(N, T^{[d]})$ be a minimal idempotent with $ux = x$. By the above argument, u is also a minimal idempotent of $E(N, \mathcal{G}^{[d]})$, whence $N = \overline{\mathcal{O}(A, \mathcal{F}^{[d]})} = \overline{\mathcal{O}(x, \mathcal{G}^{[d]})}$ is $\mathcal{G}^{[d]}$-minimal.

Finally, we show $\mathcal{F}^{[d]}$-minimal points are dense in N. Let $B \subseteq N$ be an $\mathcal{F}^{[d]}$-minimal subset. Then $\mathcal{O}(B, T^{[d]}) = \bigcup\{(T^{[d]})^n B : n \in \mathbb{Z}\}$ is a $\mathcal{G}^{[d]}$-invariant subset of N. Since $(N, \mathcal{G}^{[d]})$ is minimal, $\mathcal{O}(B, T^{[d]})$ is dense in N. Note that every point in $\mathcal{O}(B, T^{[d]})$ is $\mathcal{F}^{[d]}$-minimal, hence the proof is completed. $\square$

Setting $A = \Delta^{[d]}$ we have

Corollary 4.9. *Let (X, T) be a minimal system and let $d \geq 1$ be an integer. Then $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system, and $\mathcal{F}^{[d]}$-minimal points are dense in $\mathbf{Q}^{[d]}$.*

5. Proof of Theorem 3.1-(2)

In this section we prove Theorem 3.1-(2). That is, we show that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$ for all $x \in X$.

5.1. **A useful lemma.** The following lemma is a key step to show the uniqueness of minimal sets in $\mathbf{Q}^{[d]}[x]$ for $x \in X$. Unlike the case when (X, T) is minimal distal, we need to use the enveloping semigroup theory.

Lemma 5.1. *Let (X, T) be a minimal system and let $d \geq 1$ be an integer. If $(x^{[d-1]}, \mathbf{w}) \in \mathbf{Q}^{[d]}(X)$ for some $\mathbf{w} \in X^{[d-1]}$ and it is $\mathcal{F}^{[d]}$-minimal, then*

$$(x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

Proof. Since $(x^{[d-1]}, \mathbf{w}) \in \mathbf{Q}^{[d]}(X)$ and $(\mathbf{Q}^{[d]}, \mathcal{G}^{[d]})$ is a minimal system by Corollary 4.9, $(x^{[d-1]}, \mathbf{w})$ is in the $\mathcal{G}^{[d]}$-orbit closure of $x^{[d]}$, i.e. there are sequences $\{n_k\}_k, \{n_k^1\}_k, \ldots, \{n_k^d\}_k \subseteq \mathbb{Z}$ such that

$$(T_d^{[d]})^{n_k} (T_1^{[d]})^{n_k^1} \ldots (T_{d-1}^{[d]})^{n_k^{d-1}} (T^{[d]})^{n_k^d} (x^{[d-1]}, x^{[d-1]}) \to (x^{[d-1]}, \mathbf{w}),\ k \to \infty.$$

Let

$$\mathbf{a_k} = (T_1^{[d-1]})^{n_k^1} \ldots (T_{d-1}^{[d-1]})^{n_k^{d-1}} (T^{[d-1]})^{n_k^d} (x^{[d-1]}),$$

then the above limit can be rewritten as

$$(T_d^{[d]})^{n_k} (\mathbf{a_k}, \mathbf{a_k}) = (\mathrm{id}^{[d-1]} \times T^{[d-1]})^{n_k} (\mathbf{a_k}, \mathbf{a_k}) \to (x^{[d-1]}, \mathbf{w}),\ k \to \infty. \tag{5.1}$$

Let

$$\pi_1 : (X^{[d]}, \mathcal{F}^{[d]}) \to (X^{[d-1]}, \mathcal{F}^{[d]}), \quad (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}',$$
$$\pi_2 : (X^{[d]}, \mathcal{F}^{[d]}) \to (X^{[d-1]}, \mathcal{F}^{[d]}), \quad (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}'',$$

be projections to the first 2^{d-1} coordinates and last 2^{d-1} coordinates respectively. For π_1 we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]} \mapsto T_i^{[d-1]}$ for $1 \leq i \leq d-1$ and $T_d^{[d]} \mapsto \mathrm{id}^{[d-1]}$. For π_2 we consider the action of the group $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ via the representation $T_i^{[d]} \mapsto T_i^{[d-1]}$ for $1 \leq i \leq d-1$ and $T_d^{[d]} \mapsto T^{[d-1]}$.

Denote the corresponding semigroup homomorphisms of enveloping semigroups by

$$\pi_1^* : E(X^{[d]}, \mathcal{F}^{[d]}) \to E(X^{[d-1]}, \mathcal{F}^{[d]}), \quad \pi_2^* : E(X^{[d]}, \mathcal{F}^{[d]}) \to E(X^{[d-1]}, \mathcal{F}^{[d]}).$$

Notice that for this action of $\mathcal{F}^{[d]}$ on $X^{[d-1]}$ clearly

$$\pi_1^*(E(X^{[d]}, \mathcal{F}^{[d]})) = E(X^{[d-1]}, \mathcal{F}^{[d-1]}) \text{ and } \pi_2^*(E(X^{[d]}, \mathcal{F}^{[d]})) = E(X^{[d-1]}, \mathcal{G}^{[d-1]})$$

as subsets of $(X^{[d-1]})^{X^{[d-1]}}$. Thus for any $p \in E(X^{[d]}, \mathcal{F}^{[d]})$ and $\mathbf{x} \in X^{[d]}$, we have

$$p\mathbf{x} = p(\mathbf{x}', \mathbf{x}'') = (\pi_1^*(p)\mathbf{x}', \pi_2^*(p)\mathbf{x}'').$$

Now fix a minimal left ideal $\mathbf{L}$ of $E(X^{[d]}, \mathcal{F}^{[d]})$. By (5.1), $\mathbf{a_k} \to x^{[d-1]}, k \to \infty$. Since $(\mathbf{Q}^{[d-1]}(X), \mathcal{G}^{[d-1]})$ is minimal, there exists $p_k \in \mathbf{L}$ such that $\mathbf{a_k} = \pi_2^*(p_k)x^{[d-1]}$. Without loss of generality, we assume that $p_k \to p \in \mathbf{L}$. Then

$$\pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k} \to x^{[d-1]} \text{ and } \pi_2^*(p_k)x^{[d-1]} \to \pi_2^*(p)x^{[d-1]}.$$

Hence

$$\pi_2^*(p)x^{[d-1]} = x^{[d-1]}. \tag{5.2}$$

Since $\mathbf{L}$ is a minimal left ideal and $p \in \mathbf{L}$, by Proposition A.1 there exists a minimal idempotent $v \in J(\mathbf{L})$ such that $vp = p$. Then we have

$$\pi_2^*(v)x^{[d-1]} = \pi_2^*(v)\pi_2^*(p)x^{[d-1]} = \pi_2^*(vp)x^{[d-1]} = \pi_2^*(p)x^{[d-1]} = x^{[d-1]}.$$

Let

$$F = \mathfrak{G}(\overline{\mathcal{F}^{[d-1]}}(x^{[d-1]}), x^{[d-1]}) = \{\alpha \in v\mathbf{L} : \pi_2^*(\alpha)x^{[d-1]} = x^{[d-1]}\}$$

be the Ellis group. Then F is a subgroup of the group $v\mathbf{L}$. By (5.2), we have that $p \in F$.

Since F is a group and $p \in F$. We have

$$pFx^{[d]} = Fx^{[d]} \subset \pi_2^{-1}(x^{[d-1]}). \tag{5.3}$$

Since $vx^{[d]} \in Fx^{[d]}$, there is some $\mathbf{x_0} \in Fx^{[d]}$ such that $vx^{[d]} = p\mathbf{x_0}$. Set $\mathbf{x_k} = p_k\mathbf{x_0}$. Then

$$\mathbf{x_k} = p_k\mathbf{x_0} \to p\mathbf{x_0} = vx^{[d]} = (\pi_1^*(v)x^{[d-1]}, x^{[d-1]}),\ k \to \infty,$$

and

$$\pi_2(\mathbf{x_k}) = \pi_2(p_k\mathbf{x_0}) = \pi_2^*(p_k)x^{[d-1]} = \mathbf{a_k} \to x^{[d-1]},\ k \to \infty.$$

Let $\mathbf{x_k} = (\mathbf{b_k}, \mathbf{a_k}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$. Then $\lim_k \mathbf{b_k} = \pi_1^*(v)x^{[d-1]}$.

By (5.1), we have $(T^{[d-1]})^{n_k}\mathbf{a_k} \to \mathbf{w}, k \to \infty$. Hence

$$(\mathrm{id}^{[d-1]} \times T^{[d-1]})^{n_k}(\mathbf{b_k}, \mathbf{a_k}) = (\mathbf{b_k}, (T^{[d-1]})^{n_k}\mathbf{a_k}) \to (\pi_1^*(v)x^{[d-1]}, \mathbf{w}),\ k \to \infty. \tag{5.4}$$

Since $\mathrm{id}^{[d-1]} \times T^{[d-1]} = T_d^{[d]} \in \mathcal{F}^{[d]}$ and $(\mathbf{b_k}, \mathbf{a_k}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, we have

$$(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}). \tag{5.5}$$

Since $(x^{[d-1]}, \mathbf{w})$ is $\mathcal{F}^{[d]}$ minimal by assumption, by Proposition A.2 there is some minimal idempotent $u \in J(\mathbf{L})$ such that

$$u(x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) = (x^{[d-1]}, \mathbf{w}).$$

Since $u, v \in \mathbf{L}$ are minimal idempotents in the same minimal left ideal $\mathbf{L}$, we have $uv = u$ by Proposition A.1. Thus

$$\begin{aligned} u(\pi_1^*(v)x^{[d-1]}, \mathbf{w}) &= (\pi_1^*(u)\pi_1^*(v)x^{[d-1]}, \pi_2^*(u)\mathbf{w}) \\ &= (\pi_1^*(uv)x^{[d-1]}, \mathbf{w}) = (\pi_1^*(u)x^{[d-1]}, \mathbf{w}) = (x^{[d-1]}, \mathbf{w}). \end{aligned}$$

By (5.5), we have

$$(x^{[d-1]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]}).$$

The proof is completed. □

5.2. **Proof of Theorem 3.1-(2).** Let (X, T) be a system and $x \in X$. Recall

$$\mathbf{Q}^{[d]}[x] = \{\mathbf{z} \in \mathbf{Q}^{[d]}(X) : z_\emptyset = x\}.$$

With the help of Lemma 5.1 we have

Proposition 5.2. *Let (X, T) be a minimal system and let $d \geq 1$ be an integer. If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then*

$$x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x}).$$

Especially, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}^{[d]}[x]$.

Proof. It is sufficient to show the following claim:

S(d): *If $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$, then there exists a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(\mathbf{x}) \to x^{[d]}$.*

The case **S(1)** is trivial. To make the idea clearer, we show the case when $d = 2$. Let $(x, a, b, c) \in \mathbf{Q}^{[2]}(X)$. We may assume that (x, a, b, c) is $\mathcal{F}^{[2]}$-minimal, or we replace it by some $\mathcal{F}^{[2]}$-minimal point in its $\mathcal{F}^{[2]}$ orbit closure. Since (X, T) is minimal, there is a sequence $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k}a \to x$. Without loss of generality we assume $T^{n_k}c \to c'$. Then we have

$$(T_1^{[2]})^{n_k}(x, a, b, c) = (\mathrm{id} \times T \times \mathrm{id} \times T)^{n_k}(x, a, b, c) \to (x, x, b, c'),\ k \to \infty.$$

Since (x, a, b, c) is $\mathcal{F}^{[2]}$-minimal, (x, x, b, c') is also $\mathcal{F}^{[2]}$-minimal. By Lemma 5.1, $(x, x, b, c') \in \overline{\mathcal{F}^{[2]}}(x^{[2]})$. Together with $\mathrm{id} \times T \times \mathrm{id} \times T = T_1^{[2]} \in \mathcal{F}^{[2]}$ and the minimality of the system $(\overline{\mathcal{F}^{[2]}}(x^{[2]}), \mathcal{F}^{[2]})$ (Theorem 3.1-(1)), it is easy to see there exists a sequence $F_k \in \mathcal{F}^{[2]}$ such that $F_k(x, a, b, c) \to x^{[2]}$. Hence we have **S(2)**.

Now we assume **S(d)** holds for $d \geq 1$. Let $\mathbf{x} \in \mathbf{Q}^{[d+1]}[x]$. We may assume that $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, or we replace it by some $\mathcal{F}^{[d+1]}$-minimal point in its $\mathcal{F}^{[d+1]}$-orbit closure. Let $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}', \mathbf{x}'' \in X^{[d]}$. Then $\mathbf{x}' \in \mathbf{Q}^{[d]}[x]$. By **S(d)**, there is a sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k\mathbf{x}' \to x^{[d]}$. Without loss of generality, we assume that $F_k\mathbf{x}'' \to \mathbf{w}, k \to \infty$. Then

$$(F_k \times F_k)\mathbf{x} = (F_k \times F_k)(\mathbf{x}', \mathbf{x}'') \to (x^{[d]}, \mathbf{w}) \in \mathbf{Q}^{[d+1]}(X),\ k \to \infty.$$

Since $F_k \times F_k \in \mathcal{F}^{[d+1]}$ and $\mathbf{x}$ is $\mathcal{F}^{[d+1]}$-minimal, $(x^{[d]}, \mathbf{w})$ is also $\mathcal{F}^{[d+1]}$-minimal. By Lemma 5.1, $(x^{[d]}, \mathbf{w}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. Since $(\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal by Theorem 3.1-(1), we have $x^{[d+1]}$ is in the $\mathcal{F}^{[d+1]}$-orbit closure of $\mathbf{x}$. Hence we have **S(d+1)**, and the proof of claim is completed.

Since $x^{[d]} \in \overline{\mathcal{F}^{[d]}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{Q}^{[d]}[x]$ and $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal, it is easy to see that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ intersects all $\mathcal{F}^{[d]}$-minimal sets in $\mathbf{Q}^{[d]}[x]$ and hence it is the unique $\mathcal{F}^{[d]}$-minimal set in $\mathbf{Q}^{[d]}[x]$. The proof is completed. □

6. Lifting $\mathbf{RP}^{[d]}$ from factors to extensions

In this section, first we give some equivalent conditions for $\mathbf{RP}^{[d]}$, and give the proof of Theorem 3.8-(2), i.e. lifting $\mathbf{RP}^{[d]}$ from factors to extensions.

6.1. **Equivalent conditions for $\mathbf{RP}^{[d]}$.** In this subsection we collect some equivalent conditions for $\mathbf{RP}^{[d]}$.

Proposition 6.1. *Let (X,T) be a minimal system and $d \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) $(x,y) \in \mathbf{RP}^{[d]}$;
(2) $(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;
(3) $(x, x_*^{[d]}, y, x_*^{[d]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

Proof. By Theorem 3.2, we have (1) ⇔ (2). By Lemma 2.6 we have (3) ⇒ (1). Now show (2) ⇒ (3).

If (2) holds, then $(x,y,y,\dots,y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ and $(x,y) \in \mathbf{RP}^{[d]}$. Since $(x,y) \in \mathbf{RP}^{[d]} \subset \mathbf{RP}^{[d-1]}$, $(x, y_*^{[d]}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$. By Theorem 3.1, $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal. There is some sequence $F_k \in \mathcal{F}^{[d]}$ such that $F_k(x, y_*^{[d]}) \to x^{[d]},\ k \to \infty$. Then

$$F_k \times F_k(x, y_*^{[d]}, y, y_*^{[d]}) \to (x, x_*^{[d]}, y, x_*^{[d]}),\ k \to \infty.$$

Thus we have (3), and the proof is completed. □

Lemma 6.2. *Let (X,T) be a minimal system. Then $(x,y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$.*

Proof. If $(x,y) \in \mathbf{RP}^{[d]}$, then by Proposition 6.1, we have $(x, x_*^{[d]}, y, x_*^{[d]}) = (x^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$.

Conversely, assume that $(x,x,\dots,x,y) \in \mathbf{Q}^{[d+1]}$. Since $\mathbf{Q}^{[d+1]}$ is invariant under the Euclidean permutation of $X^{[d+1]}$, we have $(x, x_*^{[d]}, y, x_*^{[d]}) \in \mathbf{Q}^{[d+1]}$. This means that $(x,y) \in \mathbf{RP}^{[d]}$ by Lemma 2.6. □

6.2. **Lifting $\mathbf{RP}^{[d]}$ from factors to extensions.** In this section we will show Theorem 3.8-(2). First we need a lemma.

Lemma 6.3. *Let $\pi : (X,T) \to (Y,T)$ be an extension of minimal systems. If $(y_1,y_2) \in \mathbf{P}(Y,T)$ and $x_1 \in \pi^{-1}(y_1)$ then there exists $x_2 \in \pi^{-1}(y_2)$ such that $(x_1,x_2) \in \mathbf{P}(X,T)$.*

Proof. Since $(y_1,y_2) \in \mathbf{P}(Y,T)$, by Proposition A.3 there is an minimal idempotent $u \in E(X,T)$ such that $uy_1 = uy_2 = y_2$. Let $x_2 = ux_1$, then $\pi(x_2) = uy_1 = y_2$. By Proposition A.3 $(x_1,x_2) \in \mathbf{P}(X,T)$ and $\pi \times \pi(x_1,x_2) = (y_1,y_2)$. □

Theorem 6.4. *Let $\pi : (X,T) \to (Y,T)$ be an extension of minimal systems. If $(y_1,y_2) \in \mathbf{RP}^{[d]}(Y)$, then there is $(z_1,z_2) \in \mathbf{RP}^{[d]}(X)$ such that*

$$\pi \times \pi(z_1,z_2) = (y_1,y_2).$$

Proof. First we claim that it is sufficient to show the result when (y_1,y_2) is a minimal point of $(Y \times Y, T \times T)$. As a matter of fact, by Proposition A.3 there is a minimal point $(y_1',y_2') \in \overline{\mathcal{O}((y_1,y_2),T\times T)}$ such that (y_1',y_2') is proximal to (y_1,y_2). Now (y_1',y_2') is minimal and $(y_1',y_2') \in \mathbf{RP}^{[d]}(Y)$. If we have the claim already, then there is $(x_1',x_2') \in \mathbf{RP}^{[d]}(X)$ with $\pi\times\pi(x_1',x_2') = (y_1',y_2')$. Since $(y_1,y_1'),(y_2,y_2') \in \mathbf{P}(Y,T)$, then by Lemma 6.3 there are $x_1,x_2 \in X$ with $\pi\times\pi(x_1,x_2) = (y_1,y_2)$ such that (x_1',x_1), $(x_2',x_2) \in \mathbf{P}(X,T)$. This implies that $(x_1,x_2) \in \mathbf{RP}^{[d]}(X)$ by Theorem 3.3. Hence we have the result for general case.

So we may assume that (y_1,y_2) is a minimal point of $(Y\times Y, T\times T)$. To make the idea of the proof clearer, we show the case for $d=1$ first (see Figure 1). Since $(y_1,y_2) \in \mathbf{RP}^{[1]}(Y)$, by Proposition 6.1 $(y_1,y_1,y_2,y_1) \in \overline{\mathcal{F}^{[2]}}(y_1^{[2]})$. So there is some sequence $F_k \in \mathcal{F}^{[2]}$ such that

$$F_k y_1^{[2]} \to (y_1,y_1,y_2,y_1),\ k\to\infty.$$

Take a point $x_1 \in \pi^{-1}(y_1)$. Without loss of generality, we may assume that

$$F_k x_1^{[2]} \to (x_1,x_2,x_3,x_4),\ k\to\infty.$$

Then $\pi^{[2]}(x_1,x_2,x_3,x_4) = (y_1,y_1,y_2,y_1)$. Take $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k}x_2 \to x_1, k \to \infty$ and assume that $T^{n_k}x_4 \to x_4', k\to\infty$. Then

$$(\mathrm{id}\times T\times \mathrm{id}\times T)^{n_k}(x_1,x_2,x_3,x_4) \to (x_1,x_1,x_3,x_4'),\ k\to\infty.$$

Since $\mathrm{id}\times T\times\mathrm{id}\times T = T_1^{[2]} \in \mathcal{F}^{[2]}$, we have $(x_1,x_1,x_3,x_4') \in \overline{\mathcal{F}^{[2]}}(x_1^{[2]})$. Now take $\{m_k\}\subset\mathbb{Z}$ such that $T^{m_k}x_3 \to x_1, k\to\infty$ and assume that $T^{m_k}x_4' \to x_4'', k\to\infty$. Then

$$(\mathrm{id}\times\mathrm{id}\times T\times T)^{m_k}(x_1,x_1,x_3,x_4') \to (x_1,x_1,x_1,x_4''),\ k\to\infty.$$

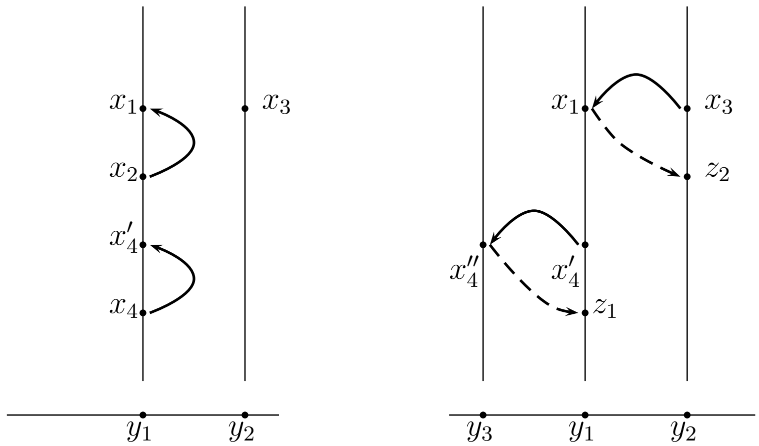
Since $\mathrm{id}\times\mathrm{id}\times T\times T = T_2^{[2]} \in \mathcal{F}^{[2]}$, we have $(x_1,x_1,x_1,x_4'') \in \overline{\mathcal{F}^{[2]}}(x_1^{[2]})$. By Lemma 6.2 $(x_1,x_4'') \in \mathbf{RP}^{[1]}(X)$. Let $y_3 = \pi(x_4'')$. Note that $(x_1,x_4'') \in \overline{\mathcal{O}((x_3,x_4'),T\times T)}$, and we have $(y_3,y_1) \in \overline{\mathcal{O}((y_1,y_2),T\times T)}$. Since (y_1,y_2) is $T\times T$-minimal, there is a sequence $\{a_k\}\subset\mathbb{Z}$ such that $(T\times T)^{a_k}(y_3,y_1) \to (y_1,y_2), k\to\infty$. Without loss of generality, we may assume that there are $z_1,z_2\in X$ such that

$$(T\times T)^{a_k}(x_4'',x_1) \to (z_1,z_2),\ k\to\infty$$

Since $(x_1,x_4'') \in \mathbf{RP}^{[1]}(X)$ and $\mathbf{RP}^{[1]}(X)$ is closed and invariant, we have $(z_1,z_2) \in \overline{\mathcal{O}((x_4'',x_1),T\times T)} \subset \mathbf{RP}^{[1]}(X)$. Note that

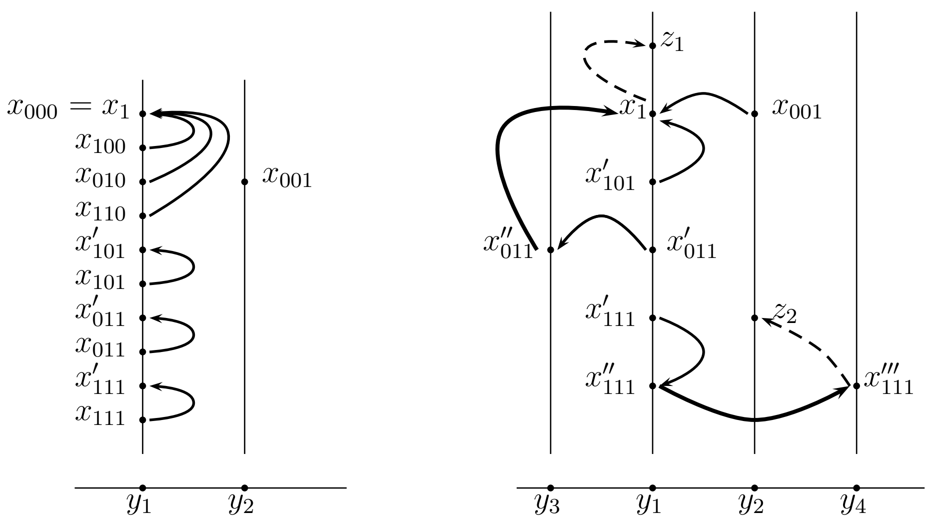
$$\pi\times\pi(z_1,z_2) = \lim_k (T\times T)^{a_k}(\pi(x_4''),\pi(x_1)) = \lim_k (T\times T)^{a_k}(y_3,y_1) = (y_1,y_2),$$

we are done for the case $d=1$. For the proof when $d=2$, see Figure 2.



$$(x_1,x_2,x_3,x_4) \downarrow (x_1,x_1,x_3,x_4') \qquad (x_1,x_1,x_3,x_4') \downarrow (x_1,x_1,x_1,x_4'')$$

Figure 1. The case $d=1$



$$(x_{000},x_{100},x_{010},x_{110},x_{001},x_{101},x_{011},x_{111}) \downarrow (x_1,x_1,x_1,x_1,x_{001},x_{101}',x_{011}',x_{111}')$$

$$(x_1,x_1,x_1,x_1,x_{001},x_{101}',x_{011}',x_{111}') \downarrow (x_1,x_1,x_1,x_1,x_1,x_1,x_{011}'',x_{111}'') \downarrow (x_1,x_1,x_1,x_1,x_1,x_1,x_1,x_{111}''')$$

Figure 2. The case $d=2$

The idea of the proof in the general case is the following. For a point $\mathbf{x}\in\overline{\mathcal{F}^{[d+1]}}(x_1)$ we apply face transformations F_1^k such that the first 2^d-coordinates of $\mathbf{x}_1=\lim F_1^k\mathbf{x}$ will be $x_1^{[d]}$. Then apply face transformations F_2^k such that the first 2^d+2^{d-1}-coordinates of $\mathbf{x}_2=\lim F_2^k\mathbf{x}_1$ will be $(x_1^{[d]},x_1^{[d-1]})$. Repeating this process we get a

point $((x_1^{[d+1]})_*, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1)$ which implies that $(x_1, x_2) \in \mathbf{RP}^{[d]}(X)$. Then we use the same idea used in the proof when $d = 1, 2$ to trace back to find (z_1, z_2). Here are the details.

Now let $(y_1, y_2) \in \mathbf{RP}^{[d]}(Y)$, then by Proposition 6.1, $(y_1^{[d]}, y_2, (y_1^{[d]})_*) \in \overline{\mathcal{F}^{[d+1]}}(y_1^{[d+1]})$. So there is some sequence $F_k \in \mathcal{F}^{[d+1]}$ such that

$$F_k y_1^{[d+1]} \to (y_1^{[d]}, y_2, (y_1^{[d]})_*),\ k \to \infty.$$

Without loss of generality, we may assume that

$$F_k x_1^{[d+1]} \to \mathbf{x},\ k \to \infty. \tag{6.1}$$

Then $x_\emptyset = x_1$ and $\pi^{[d+1]}(\mathbf{x}) = (y_1^{[d]}, y_2, (y_1^{[d]})_*)$.

Let $\mathbf{x_I} = (x_\epsilon : \epsilon(d+1) = 0) \in X^{[d]}$ and $\mathbf{x_{II}} = (x_\epsilon : \epsilon(d+1) = 1) \in X^{[d]}$. Then $\mathbf{x} = (\mathbf{x_I}, \mathbf{x_{II}})$. Note that

$$\pi^{[d]}(\mathbf{x_I}) = \pi^{[d]}(x_1^{[d]}) = y_1^{[d]},\ \text{and}\ \pi^{[d]}(\mathbf{x_{II}}) = (y_2, (y_1^{[d]})_*).$$

By Proposition 5.2, there is some sequence $F_k^1 \in \mathcal{F}^{[d]}$ such that

$$F_k^1(\mathbf{x_I}) \to x_1^{[d]},\ k \to \infty.$$

We may assume that

$$F_k^1(\mathbf{x_{II}}) \to \mathbf{x'_{II}},\ k \to \infty.$$

Note that $\pi^{[d]}(\mathbf{x_{II}}) = \pi^{[d]}(\mathbf{x'_{II}}) = (y_2, (y_1^{[d]})_*)$.

Let $F_k^1 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$. Let $H_k^1 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d+1) = 1) = F_k^1.$$

Then

$$H_k^1(\mathbf{x}) = F_k^1 \times F_k^1(\mathbf{x_I}, \mathbf{x_{II}}) \to (x_1^{[d]}, \mathbf{x'_{II}}) \triangleq \mathbf{x^1} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}),\ k \to \infty.$$

Let $\mathbf{y^1} = \pi^{[d+1]}(\mathbf{x^1})$. It is easy to see that $x_\epsilon^1 = x_1$ if $\epsilon(d+1) = 0$. For $\mathbf{y^1}$, $y^1_{\{d+1\}} = y^1_{00\ldots01} = y_2$ and $y_\epsilon^1 = y_1$ for all $\epsilon \neq \{d+1\}$.

Let $\mathbf{x_I^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) \in X^{[d]}$ and $\mathbf{x_{II}^1} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F_k^2 \in \mathcal{F}^{[d]}$ such that

$$F_k^2(\mathbf{x_I^1}) \to x_1^{[d]},\ F_k^2(\mathbf{x_{II}^1}) \to \mathbf{x_{II}^1}', k \to \infty$$

and $\pi^{[d]}(\mathbf{x_{II}^1}') = (y_1^{[d-1]}, y_3, (y_1^{[d-1]})_*)$ for some $y_3 \in Y$.

Let $F_k^2 = (S_{\epsilon'}^k : \epsilon' \in \{0,1\}^d)$. Let $H_k^2 = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that

$$(S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 0) = (S_\epsilon^k : \epsilon \in \{0,1\}^{d+1}, \epsilon(d) = 1) = F_k^2.$$

Then let

$$H_k^2(\mathbf{x^1}) \to \mathbf{x^2} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}),\ k \to \infty.$$

Let $\mathbf{y^2} = \pi^{[d+1]}(\mathbf{x^2})$. Then $H_k^2(\mathbf{y^1}) \to \mathbf{y^2},\ k \to \infty$. From this one has that $(y_3, y_1) \in \overline{\mathcal{O}((y_1, y_2), T \times T)}$. By the definition of $\mathbf{x^2}, \mathbf{y^2}$, it is easy to see that $x_\epsilon^2 = x_1$ if $\epsilon(d+1) = 0$ or $\epsilon(d) = 0$; $y^2_{\{d,d+1\}} = y^2_{00\ldots011} = y_3$ and $y_\epsilon^2 = y_1$ for all $\epsilon \neq \{d, d+1\}$.

Now assume that we have $\mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ for $1 \le j \le d$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$ such that $x_\epsilon^j = x_1$ if there exists some k with $d - j + 2 \le k \le d + 1$ such that

$\epsilon(k) = 0$; $y^j_{\{d-j+2,\dots,d,d+1\}} = y_{j+1}$ and $y^j_\epsilon = y_1$ for all $\epsilon \neq \{d-j+2,\dots,d,d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1,y_j), T\times T)}$.

Let $\mathbf{x^j_I} = (x_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) \in X^{[d]}$ and $\mathbf{x^j_{II}} = (x_\epsilon : \epsilon \in \{0,1\}^{d-j+1}, \epsilon(d-j+1) = 1) \in X^{[d]}$. By Proposition 5.2, there is some sequence $F^{j+1}_k \in \mathcal{F}^{[d]}$ such that

$$F^{j+1}_k(\mathbf{x^j_I}) \to x_1^{[d]},\ F^{j+1}_k(\mathbf{x^j_{II}}) \to \mathbf{x^j_{II}}', k\to\infty.$$

Let $F^{j+1}_k = (S^k_{\epsilon'} : \epsilon' \in \{0,1\}^d)$. Let $H^{j+1}_k = (S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}) \in \mathcal{F}^{[d+1]}$ such that $(S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 0) = (S^k_\epsilon : \epsilon \in \{0,1\}^{d+1}, \epsilon(d-j+1) = 1) = F^{j+1}_k$. Then let

$$H^{j+1}_k(\mathbf{x^j}) \to \mathbf{x^{j+1}} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}),\ k\to\infty.$$

It is easy to see that $x^{j+1}_\epsilon = x_1$ if there exists some k with $d-j+1 \le k \le d+1$ such that $\epsilon(k) = 0$.

Let $\mathbf{y^{j+1}} = \pi^{[d+1]}(\mathbf{x^{j+1}})$. Then $y^{j+1}_\epsilon = y_1$ for all $\epsilon \neq \{d-j+1, d-j+2,\dots,d+1\}$, and denote $y^j_{\{d-j+1,d-j+2,\dots,d+1\}} = y_{j+2}$. Note that $H^2_k(\mathbf{y^j}) \to \mathbf{y^{j+1}},\ k\to\infty$. From this one has that $(y_{j+2}, y_1) \in \overline{\mathcal{O}((y_1,y_{j+1}), T\times T)}$.

Inductively we get $\mathbf{x^1},\dots,\mathbf{x^{d+1}}$ and $\mathbf{y^1},\dots,\mathbf{y^{d+1}}$ such that for all $1\le j\le d+1$ $\mathbf{x^j} \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]})$ with $\pi^{[d+1]}(\mathbf{x^j}) = \mathbf{y^j}$. And $x^j_\epsilon = x_1$ if there exists some k with $d-j+2 \le k \le d+1$ such that $\epsilon(k) = 0$; $y^j_{\{d-j+2,\dots,d,d+1\}} = y_{j+1}$ and $y^j_\epsilon = y_1$ for all $\epsilon \neq \{d-j+2,\dots,d,d+1\}$, and $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1,y_j), T\times T)}$.

For $\mathbf{x^{d+1}}$, we have that $x^{d+1}_\epsilon = x_1$ if there exists some k with $1\le k\le d+1$ such that $\epsilon(k) = 0$. That means there is some $x_2 \in X$ such that

$$\mathbf{x^{d+1}} = (x_1, x_1, \dots, x_1, x_2) \in \overline{\mathcal{F}^{[d+1]}}(x_1^{[d+1]}).$$

By Lemma 6.2, $(x_1,x_2) \in \mathbf{RP}^{[d]}(X)$. Note that $\pi(x_2) = y_{d+2}$.

Since $(y_{j+1}, y_1) \in \overline{\mathcal{O}((y_1,y_j), T\times T)}$ for all $1\le j\le d+1$, we have $(y_{d+2}, y_1) \in \overline{\mathcal{O}((y_1,y_2), T\times T)}$ or $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1,y_2), T\times T)}$. Without loss of generality, we assume that $(y_1, y_{d+2}) \in \overline{\mathcal{O}((y_1,y_2), T\times T)}$. Since (y_1,y_2) is $T\times T$-minimal, there is some $\{n_k\} \subset \mathbb{Z}$ such that $(T\times T)^{n_k}(y_1, y_{d+2}) \to (y_1,y_2), k\to\infty$. Without loss of generality, we assume that

$$(T\times T)^{n_k}(x_1,x_2) \to (z_1,z_2),\ k\to\infty.$$

Since $\mathbf{RP}^{[d]}(X)$ is closed and invariant, we have

$$(z_1,z_2) \in \overline{\mathcal{O}((x_1,x_2), T\times T)} \subset \mathbf{RP}^{[d]}(X).$$

And

$$\pi\times\pi(z_1,z_2) = \lim_k (T\times T)^{n_k}(\pi(x_1),\pi(x_2)) = \lim_k (T\times T)^{n_k}(y_1, y_{d+2}) = (y_1,y_2).$$

The proof is completed. □

7. A combinatorial consequence and group actions

7.1. **A combinatorial consequence.** We have the following combinatorial consequence of the fact that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal.

Proposition 7.1. *Let (X,T) be a minimal system, $x \in X$ and U be an open neighborhood of x. Put $S = \{n \in \mathbb{Z} : T^n x \in U\}$. Then for each $d \geq 1$,*

$$\{(n_1, \dots, n_d) \in \mathbb{Z}^d : n_1\epsilon_1 + \cdots + n_d\epsilon_d \in S, \epsilon_i \in \{0,1\}, 1 \leq i \leq d\}$$

is syndetic.

Proof. This follows by that fact that $x^{[d]}$ is a minimal point under the face group action $\mathcal{F}^{[d]}$. □

To understand S better we show the following proposition which is similar to [24, Proposition 2.3]. Note that a collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ is a *family* if it is upwards, i.e. $A \in \mathcal{F}$ and $A \subset B$ imply that $B \in \mathcal{F}$.

Proposition 7.2. *The family of dynamically syndetic subsets is the family generated by the sets S whose indicator functions 1_S are the minimal points of $(\{0,1\}^{\mathbb{Z}}, \sigma)$ and $0 \in S$, where σ is the shift.*

Proof. Put $\Sigma = \{0,1\}^{\mathbb{Z}}$. We denote the family generated by the sets containing $\{0\}$ whose indicator functions are the minimal points of (Σ, σ) by $\mathcal{F}_m$. Clearly, if 1_F is the indicator function of F then $F = N(1_F, [1])$, where $[1] = \{s \in \Sigma : s(0) = 1\}$. Hence $\mathcal{F}_m$ is contained in the family of dynamical syndetic subsets.

On the other hand, let A be a dynamical syndetic subset. Then there exist a minimal system (X,T) with metric d, $x \in X$ and an open neighborhood V of x such that $A \supset N(x,V) = \{n \in \mathbb{Z} : T^n x \in V\}$. It is easy to see that we can shrink V to an open neighborhood V' of x whose boundary is disjoint from the orbit of x.

Then do the classical lifting trick, a la Glasner, Adler etc. Let

$$Y = \{(z,t) \in X \times \Sigma : t(i) = 1 \text{ implies } T^i z \in \mathrm{cl}(V') \text{ and } t(i) = 0 \text{ implies } T^i z \in \mathrm{cl}(X \setminus V')\}$$

Then Y is a $T \times \sigma$-invariant closed subset of $X \times \Sigma$. Since the orbit of x doesn't meet the boundary of V', there is a unique $t \in \Sigma$ such that $(x,t) \in Y$ and t is the indicator function of $N(x,V')$. Take a minimal subset J of $(Y, T \times \sigma)$ with $J \subset \overline{\mathcal{O}((x,t), T \times \sigma)}$ and let $\pi_X : J \to X$ be the projective map. Since (X,T) is minimal, $\pi_X(J) = X$. Hence $(x,t) \in J$. Projecting J to Σ we see that t is a minimal point. Hence $A \in \mathcal{F}_m$ as $A \supset N(x,V')$ and $t = 1_{N(x,V')}$. □

Remark 7.3. We note that if S is a syndetic subset of $\mathbb{Z}$ then $S - S \supset S_1 - S_1$ for some dynamically syndetic subset S_1.

7.2. **Abelian group actions.**

Definition 7.4. Let X be a compact metric space, G be an abelian topological group acting on X and let $d \geq 1$ be an integer. A pair $(x,y) \in X \times X$ ia said to be *regionally proximal of order d of G-action* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in G^d$ such that $\rho(x,x') < \delta, \rho(y,y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \text{ for any nonempty } \epsilon \subset [d],$$

where $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$. The set of regionally proximal pairs of order d of G-action is denoted by $\mathbf{RP}_G^{[d]}(X)$, which is called *the regionally proximal relation of order d of G-action*.

A subset $S \subseteq G$ is a *central set* if there exists a system (X, G), a point $x \in X$ and a minimal point y proximal to x, and a neighborhood U_y of y such that $N(x, U_y) \subset S$. The notion of IP-set can be defined in this setting too. By the proof of Furstenberg [11, Proposition 8.10.] we have

Lemma 7.5. *Let G be an abelian group. Then any central set is an IP-set.*

So we have

Lemma 7.6. *If (X, G) is minimal, then $\mathbf{P}(X) \subset \mathbf{RP}_G^{[d]}(X)$.*

At the same time the notions of face group and parallelepiped group can be defined. So we have the following theorem by our proof

Theorem 7.7. *Let (X, G) a minimal system with G being abelian. Then $\mathbf{RP}_G^{[d]}(X)$ is a closed invariant equivalence relation. So $(X/\mathbf{RP}_G^{[d]}(X), G)$ is distal.*

Similar to [22] we may define

Definition 7.8. Let (X, G) a minimal system with G being abelian. We call $(X/\mathbf{RP}_G^{[d]}(X), G)$ the d-step nilfactor for G-action.

We think that it will be interesting to study the properties of $(X/\mathbf{RP}_G^{[d]}(X), G)$ or more general group actions.

Appendix A. Basic facts about abstract topological dynamics

In this appendix we recall some basic definitions and results in abstract topological systems. For more details, see [1, 4, 13, 16, 28, 29].

A.1. **Topological transformation groups.** A *topological dynamical systems* is a triple $\mathcal{X} = (X, \mathcal{T}, \Pi)$, where X is a compact T_2 space, $\mathcal{T}$ is a T_2 topological group and $\Pi : T \times X \to X$ is a continuous map such that $\Pi(e, x) = x$ and $\Pi(s, \Pi(t, x)) = \Pi(st, x)$. We shall fix $\mathcal{T}$ and suppress the action symbol. In lots of literatures, $\mathcal{X}$ is also called a *topological transformation group* or a *flow*. Usually we omit Π and denote a system by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a system and $x \in X$, then $\mathcal{O}(x, \mathcal{T})$ denotes the *orbit* of x, which is also denoted by $\mathcal{T}x$. A subset $A \subseteq X$ is called *invariant* if $ta \subseteq A$ for all $a \in A$ and $t \in \mathcal{T}$. When $Y \subseteq X$ is a closed and $\mathcal{T}$-invariant subset of the system $(X, \mathcal{T})$ we say that the system $(Y, \mathcal{T})$ is a *subsystem* of $(X, \mathcal{T})$. If $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ are two dynamical systems their *product system* is the system $(X \times Y, \mathcal{T})$, where $t(x, y) = (tx, ty)$.

A system $(X, \mathcal{T})$ is called *minimal* if X contains no proper closed invariant subsets. $(X, \mathcal{T})$ is called *transitive* if every invariant open subset of X is dense. An example of an transitive system is a *point-transitive* system, which is a system with a dense

orbit. It is easy to verify that a system is minimal iff every orbit is dense. The system $(X, \mathcal{T})$ is *weakly mixing* if the product system $(X \times X, \mathcal{T})$ is transitive.

A *homomorphism* (or *extension*) of systems $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ is a continuous onto map of the phase spaces such that $\pi(tx) = t\pi(x)$ for all $t \in \mathcal{T}, x \in X$. In this case one says that $(Y, \mathcal{T})$ is a *factor* of $(X, \mathcal{T})$ and also that $(X, \mathcal{T})$ is an *extension* of $(Y, \mathcal{T})$. Define

$$R_\pi = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\},$$

then $Y = X/R_\pi$. For $n \geq 2$, define

$$R_\pi^n = \{(x_1, x_2, \dots, x_n) \in X^n : \pi(x_1) = \pi(x_2) = \dots = \pi(x_n)\},$$

and let $R_\pi^1 = X$.

A.2. **Enveloping semigroups.** Given a system $(X, \mathcal{T})$ its *enveloping semigroup* or *Ellis semigroup* $E(X, \mathcal{T})$ is defined as the closure of the set $\{t : t \in \mathcal{T}\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). For an enveloping semigroup, $E \to E : p \mapsto pq$ and $p \mapsto tp$ is continuous for all $q \in E$ and $t \in \mathcal{T}$. Note that $(X^X, \mathcal{T})$ is a system and $(E(X, \mathcal{T}), \mathcal{T})$ is its subsystem.

Let $(X, \mathcal{T}), (Y, \mathcal{T})$ be systems and $\pi : X \to Y$ be an extension. Then there is a unique continuous semigroup homomorphism $\pi^* : E(X, \mathcal{T}) \to E(Y, \mathcal{T})$ such that $\pi(px) = \pi^*(p)\pi(x)$ for all $x \in X, p \in E(X, \mathcal{T})$. When there is no confusion, we usually regard the enveloping semigroup of X as acting on Y: $p\pi(x) = \pi(px)$ for $x \in X$ and $p \in E(X, \mathcal{T})$.

A.3. **Idempotents and ideals.** For a semigroup the element u with $u^2 = u$ is called an *idempotent*. Ellis-Namakura Theorem says that for any enveloping semigroup E the set $J(E)$ of idempotents of E is not empty [4]. A non-empty subset $I \subset E$ is a *left ideal* (resp. *right ideal*) if it $EI \subseteq I$ (resp. $IE \subseteq I$). A *minimal left ideal* is the left ideal that does not contain any proper left ideal of E. Obviously every left ideal is a semigroup and every left ideal contains some minimal left ideal.

We can introduce a quasi-order (a reflexive, transitive relation) $<_L$ on the set $J(E)$ by defining $v <_L u$ if and only if $vu = v$. If $v <_L u$ and $u <_L v$ we say that u and v are *equivalent* and write $u \sim_L v$. Similarly, we define $<_R$ and $\sim_R$. An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $v <_L u$ implies $u <_L v$. The following results are well-known [5, 12]: let L be a left ideal of enveloping semigroup E and $u \in J(E)$. Then there is some idempotent v in Lu such that $v <_R u$ and $v <_L u$; an idempotent is minimal if and only if it is contained in some minimal left ideal.

Minimal left ideals have very rich algebraic properties. For example,

Proposition A.1. *Let I be a minimal left ideal, then*

(1) $I = \bigcup_{u \in J(I)} uI$ *is its partition and every uI is a group with identity $u \in J(I)$.*
(2) *All minimal idempotents in the same minimal left ideal are equivalent to each other, i.e. for all $u, v \in J(I)$, $u \sim_L v$.*

Let $(X,\mathcal{T})$ be a system and $E(X,\mathcal{T})$ be its enveloping semigroup. A subset $I \subseteq E(X,\mathcal{T})$ is a closed left ideal of $E(X,\mathcal{T})$ iff $(I,\mathcal{T})$ is a subsystem of $(E(X,\mathcal{T}),\mathcal{T})$. And I is a minimal left ideal of $E(X,\mathcal{T})$ iff $(I,\mathcal{T})$ is minimal. Let $I \subset E(X,\mathcal{T})$ be a minimal left ideal. Then for all $x \in X$, $Ix = \{px : p \in I\}$ is a minimal subset of X. Especially if $(X,\mathcal{T})$ is minimal itself, then $X = Ix$ for all $x \in X$. It follows that

Proposition A.2. *A point $x \in X$ is minimal if and only if $ux = x$ for some $u \in I$.*

A.4. **Universal point transitive system and universal minimal system.** For fixed $\mathcal{T}$, there exists a universal point-transitive system $\mathcal{S}_\mathcal{T} = (S_\mathcal{T},\mathcal{T})$ such that $\mathcal{T}$ can densely and equivariantly be embedded in $S_\mathcal{T}$. The multiplication on $\mathcal{T}$ can be extended to a multiplication on $S_\mathcal{T}$, then $S_\mathcal{T}$ is a closed semigroup with continuous right translations. The universal minimal system $\mathfrak{M} = (\mathbf{M},\mathcal{T})$ is isomorphic to any minimal left ideal in $S_\mathcal{T}$ and $\mathbf{M}$ is a closed semigroup with continuous right translations. Hence $J = J(\mathbf{M})$ of idempotents in $\mathbf{M}$ is nonempty. Moreover, $\{v\mathbf{M} : v \in J\}$ is a partition of $\mathbf{M}$ and every $v\mathbf{M}$ is a group with unit element v. Sometimes if there are chances being confusion then we will use $\mathbf{M}_\mathcal{T}$ instead of $\mathbf{M}$.

The sets $S_\mathcal{T}$ and $\mathbf{M}$ act on X as semigroups and $S_\mathcal{T} x = \overline{\mathcal{T}x}$, while for a minimal system $(X,\mathcal{T})$ we have $\mathbf{M}x = \overline{\mathcal{T}x} = X$ for every $x \in X$. A necessary and sufficient condition for x to be minimal is that $ux = x$ for some $u \in J$.

A.5. **All kinds of extensions.** Two points x_1 and x_2 are called *proximal* iff

$$\overline{\mathcal{T}(x_1,x_2)} \cap \Delta_X \neq \emptyset.$$

Let $\mathcal{U}_X$ be the unique uniform structure of X, then

$$\mathbf{P} = \mathbf{P}(X,\mathcal{T}) = \bigcap \{\mathcal{T}\alpha : \alpha \in \mathcal{U}_X\}$$

is the collection of proximal pairs in X, the *proximal relation*.

Proposition A.3. *Let $(X,\mathcal{T})$ be a system. Then*

1. *x_1, x_2 are proximal in $(X,\mathcal{T})$ iff $px_1 = px_2$ for some $p \in E(X,\mathcal{T})$.*
2. *If $x \in X$ and u is an idempotent in $E(X,\mathcal{T})$, then $(x,ux) \in \mathbf{P}$.*
3. *If $x \in X$, then there is an minimal point $x' \in \overline{\mathcal{O}(x,\mathcal{T})}$ such that $(x,x') \in \mathbf{P}$.*
4. *If (X,T) is minimal, then $(x,y) \in \mathbf{P}$ if and only if there is some minimal idempotent $u \in E(X,\mathcal{T})$ such that $y = ux$.*

The extension $\pi : (X,\mathcal{T}) \to (Y,\mathcal{T})$ is called *proximal* iff $R_\pi \subseteq \mathbf{P}$ iff $\mathbf{P}_\pi = \bigcap\{\mathcal{T}\alpha \cap R_\pi : \alpha \in \mathcal{U}_X\} = R_\pi$. π is *distal* if $\mathbf{P}_\pi = \Delta_X$. π is a *highly proximal* (HP) extension if for every closed subset A of X with $\pi(A) = Y$, necessarily $A = X$. It is easy to see that a HP extension is proximal. In the metric case an extension $\pi : (X,T) \to (Y,T)$ of minimal systems is HP iff it is an *almost 1-1 extension*, that is the set $\{y \in Y : \pi^{-1}(y) \text{ is a singleton }\}$ is a dense G_δ subset of Y.

An extension $\pi : X \to Y$ of systems is called *equicontinuous* or *almost periodic* if for every $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ such that $\mathcal{T}\alpha \cap R_\pi \subseteq \beta$.

In the metric case an equicontinuous extension is also called an *isometric extension*. The extension π is a *weakly mixing extension* when $(R_\pi,\mathcal{T})$ as a subsystem of the product system $(X \times X,\mathcal{T})$ is transitive.

A.6. **Vietoris topology and circle operation.** Let 2^X be the collection of nonempty closed subsets of X endowed with the Vietoris topology. Note that a base for the Vietoris topology on 2^X is formed by the sets

$$< U_1, U_2, \cdots, U_n >= \{A \in 2^X : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for every } i\},$$

where U_i is open in X. Then $(2^X, \mathcal{T})$ defined by $tA = \{ta : a \in A\}$ is a system again, and $S_{\mathcal{T}}$ acts on 2^X too. To avoid ambiguity we denote the action of $S_{\mathcal{T}}$ on 2^X by the *circle operation* as follows. Let $p \in S_{\mathcal{T}}$ and $D \in 2^X$, then define $p \circ D = \lim_{2^X} t_i D$ for any net $\{t_i\}_i$ in $\mathcal{T}$ with $t_i \to p$. Moreover

$$p \circ D = \{x \in X : \text{there are } d_i \in D \text{ with } x = \lim_i t_i d_i\}$$

for any net $t_i \to p$ in $S_{\mathcal{T}}$. We always have $pD \subseteq p \circ D$.

A.7. **Ellis group.** The group of automorphisms of $(\mathbf{M}, \mathcal{T})$, $G = \mathrm{Aut}(\mathbf{M}, \mathcal{T})$ can be identified with any one of the groups $u\mathbf{M}$ $(u \in J)$ as follows: with $\alpha \in uM$ we associate the automorphism $\hat{\alpha} : (\mathbf{M}, \mathcal{T}) \to (\mathbf{M}, \mathcal{T})$ given by right multiplication $\hat{\alpha}(p) = p\alpha, p \in \mathbf{M}$. The group G plays a central role in the algebraic theory. It carries a natural T_1 compact topology, called by Ellis the *τ-topology*, which is weaker than the relative topology induced on $G = u\mathbf{M}$ as a subset of $\mathbf{M}$.

It is convenient to fix a minimal left ideal $\mathbf{M}$ in $S_{\mathcal{T}}$ and an idempotent $u \in \mathbf{M}$. As explained above we identify G with $u\mathbf{M}$ and for any subset $A \subseteq G$, *τ-topology* is determined by

$$\mathrm{cl}_\tau A = u(u \circ A) = G \cap (u \circ A).$$

Also in this way we can consider the "action" of G on every system $(X, \mathcal{T})$ via the action of $S_{\mathcal{T}}$ on X. With every minimal system (X, T) and a point $x_0 \in uX = \{x \in X : ux = x\}$ we associate a τ-closed subgroup

$$\mathfrak{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}$$

the *Ellis group* of the pointed system (X, x_0).

For a homomorphism $\pi : X \to Y$ with $\pi(x_0) = y_0$ we have

$$\mathfrak{G}(X, x_0) \subseteq \mathfrak{G}(Y, y_0).$$

It is easy to see that $u\pi^{-1}(y_0) = \mathfrak{G}(Y, y_0)x_0$.

For a τ-closed subgroup F of G the derived group $H(F) = F'$ is given by:

$$H(F) = F' = \bigcap \{\mathrm{cl}_\tau O : O \text{ is a } \tau\text{-open neighborhood of } u \text{ in } F\}.$$

$H(F)$ is a τ-closed normal subgroup of F and it is characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group. In particular, for an abelian $\mathcal{T}$, the topological group $G/H(G)$ is the *Bohr compactification* of $\mathcal{T}$.

A.8. **Structure of minimal systems.** Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a homomorphism of minimal systems with $x_0 \in X$ and $y_0 = \pi(x_0) \in Y$. We say that π is a *RIC* (relatively incontractible) extension if for every $y = py_0 \in Y$, p an element of $\mathbf{M}$,

$$\pi^{-1}(y) = p \circ u\pi^{-1}(y_0) = p \circ Fx_0,$$

where $F = \mathfrak{G}(Y, y_0)$. One can show that the extension $\pi : X \to Y$ is RIC if and only if it is open and for every $n \geq 1$ the minimal points are dense in the relation R_π^n. Note that every distal extension is RIC. It then follows that every distal extension is open.

We say that a minimal system $(X, \mathcal{T})$ is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that $(X, \mathcal{T})$ is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost one-to-one extensions (or by highly proximal extensions in the non-metric case) we get the notion of HPI *systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have I *systems*. These notions can be easily relativized and we then speak about I, HPI, and PI extensions.

Theorem A.4 (Furstenberg)**.** *A metric minimal system is distal if and only if it is an I-system.*

Theorem A.5 (Veech)**.** *A metric minimal dynamical system is point distal if and only if it is an HPI-system.*

Finally we have the structure theorem for minimal systems, which we will state in its relative form (Ellis-Glasner-Shapiro [7], Veech [28], and Glasner [13]).

Theorem A.6 (Structure theorem for minimal systems)**.** *Given a homomorphism $\pi : X \to Y$ of minimal dynamical system, there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*

$$\begin{tikzcd}
X \arrow[d, "\pi"'] & X_0 \arrow[l, "\theta_0^*"'] \arrow[d, "\pi_0"'] \arrow[dr, "\sigma_1"] & & X_1 \arrow[ll, "\theta_1^*"'] \arrow[d, "\pi_1"'] & \cdots & X_\nu \arrow[d, "\pi_\nu"'] \arrow[dr, "\sigma_{\nu+1}"] & & X_{\nu+1} \arrow[ll, "\theta_{\nu+1}^*"'] \arrow[d, "\pi_{\nu+1}"'] & \cdots & X_\eta = X_\infty \arrow[d, "\pi_\infty"'] \\
Y & Y_0 \arrow[l, "\theta_0"] & Z_1 \arrow[l, "\rho_1"] & Y_1 \arrow[l, "\theta_1"] & \cdots & Y_\nu & Z_{\nu+1} \arrow[l, "\rho_{\nu+1}"] & Y_{\nu+1} \arrow[l, "\theta_{\nu+1}"] & \cdots & Y_\eta = Y_\infty
\end{tikzcd}$$

where for each $\nu \leq \eta, \pi_\nu$ is RIC, ρ_ν is isometric, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI-system.

Appendix B. Proof of Theorem 4.3

First we need the so-called *Ellis trick* in [13]. Refer to [13, Lemma X.6.1] for the proof. See [17] for more discussions about weakly mixing extensions. Recall that $\mathbf{M}$ is the universal minimal set.

Lemma B.1 (Ellis trick). *Let F be τ closed subgroup of G acting on $\mathbf{M}$ by right multiplication, $\mathbf{M} \times F \to \mathbf{M}, (p, \alpha) \mapsto p\alpha$.*

(1) *there is a minimal idempotent $\omega \in J(\mathbf{M}) \cap \overline{F}$ such that $\overline{\omega F}$ is F-minimal.*
(2) *if V is a open subset of $\overline{wF}$, then $\mathrm{int}_\tau \mathrm{cl}_\tau (V \cap wF) \neq \emptyset$.*

Lemma B.2. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal systems and $u \in J(\mathbf{M})$ be a minimal idempotent. Let $x \in uX$, $y = \pi(x)$. Then for all $n \geq 2$, any nonempty open subset U of $\overline{u\pi^{-1}(y)}$ and any transitive point $x' = (x'_1, \cdots, x'_{n-1}) \in R_\pi^{n-1}$ with $\pi(x'_j) = y, j = 1, \cdots, n-1$, we have $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$.*

Proof. Note that we have $H(F)A = F$, where $F = \mathfrak{G}(Y, y), A = \mathfrak{G}(X, x)$, since π is weakly mixing.
Claim:

$$\{ux'\} \times \pi^{-1}(y) \subset \overline{\mathcal{T}(\{x'\} \times U)}.$$

Proof of The Claim: Set $V = \{p \in \overline{F} : px \in U\}$. Then V is a nonempty open set of $\overline{F}$ and by Ellis trick we have $\widetilde{V} = \mathrm{int}_\tau \mathrm{cl}_\tau (V \cap F) \neq \emptyset$. By the definition of $H(F)$, there exists $\alpha \in F$ such that $\alpha H(F) \subseteq \mathrm{cl}_\tau \widetilde{V}$.

Since $F = AH(F) = H(F)A$, we have

$$\begin{aligned}
\overline{\mathcal{T}(\{x'\} \times U)} &\supseteq u \circ (\{x'\} \times U) \supseteq u \circ (\{x'\} \times Vx) \\
&\supseteq \{ux'\} \times u(u \circ V)x \supseteq \{ux'\} \times u(u \circ (V \cap F))x \\
&= \{ux'\} \times \mathrm{cl}_\tau (V \cap F)x \supseteq \{ux'\} \times \mathrm{cl}_\tau \widetilde{V}x \\
&\supseteq \{ux'\} \times \alpha H(F)x = \{ux'\} \times \alpha H(F)Ax \\
&= \{ux'\} \times \alpha Fx = \{ux'\} \times Fx.
\end{aligned}$$

Since π is RIC, we have $u \circ Fx = \pi^{-1}(y)$. Hence

$$\overline{\mathcal{T}(\{x'\} \times U)} \supseteq u \circ (\{ux'\} \times Fx) = \{ux'\} \times \pi^{-1}(y).$$

This ends the proof of the claim.

Now it is easy to see that $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. Let $(x_1, x_2) \in R_\pi^n$, where $x_1 \in R_\pi^{n-1}$. Since x' is a transitive point of R_π^{n-1}, there exists a $p \in S_\mathcal{T}$ such that $px' = x_1$. Then $x_2 \in \pi^{-1}(py) = p \circ \pi^{-1}(y)$. Thus

$$(x_1, x_2) \in \{px'\} \times p \circ \pi^{-1}(y) \subseteq \overline{\mathcal{T}(\{ux'\} \times \pi^{-1}(y))} \subseteq \overline{\mathcal{T}(\{x'\} \times U)}.$$

Thus we have $R_\pi^n = \overline{\mathcal{T}(\{x'\} \times U)}$. □

Theorem B.3. *Let $\pi : (X, \mathcal{T}) \to (Y, \mathcal{T})$ be a RIC weakly mixing extension of minimal metric systems and $y \in Y$. Then for all $n \geq 1$, there exists a transitive point $(x_1, x_2, \ldots, x_n)$ of R_π^n with $x_1, x_2, \ldots, x_n \in \pi^{-1}(y)$.*

Proof. It is obvious for the case when $n = 1$, since $R_\pi^1 = X$. Now assume it is true for $n-1$ ($n \geq 2$). Fix a transitive point $x' = (x_1, x_2, \dots, x_{n-1}) \in R_\pi^{n-1}$ with $x_1, x_2, \dots, x_{n-1} \in \pi^{-1}(y)$. Assume that $y \in uY$ for some minimal idempotent $u \in J(\mathbf{M})$.

For each $\epsilon > 0$, define

$$V_\epsilon = \{x \in \overline{u\pi^{-1}(y)} : \mathcal{T}(x', x) \text{ is } \epsilon\text{-dense in } \ R_\pi^n\}.$$

It is easy to verify that V_ϵ is open. Now we show that V_ϵ is dense in $\overline{u\pi^{-1}(y)}$. For any $\Lambda \subseteq X^n, z \in X^n, \delta > 0$, $\Lambda \overset{\delta}{\sim} z$ is defined by $\rho(z, z') < \delta, \forall z' \in \Lambda$.

Now let $\{z_1, z_2, \cdots, z_n\}$ be an ϵ-net of R_π^n, i.e. for each $z \in R_\pi^n$ there is some z_j ($j \in \{1, 2, \dots, n\}$) such that $\rho(z, z_j) < \epsilon$. Let U be an open subset of $\overline{w\pi^{-1}(y)}$. By Lemma B.2, $\overline{\mathcal{T}(\{x'\} \times U)} = R_\pi^n$. So there are some open subset $U_1 \supseteq U$ and $t_1 \in \mathcal{T}$ such that $t_1(\{x'\} \times U_1) \overset{\epsilon}{\sim} z_1$. Again, by Lemma B.2, $\overline{\mathcal{T}(\{x'\} \times U_1)} = R_\pi^n$. So there are an open subset $U_2 \supseteq U_1$ and $t_2 \in \mathcal{T}$ such that $t_2(\{x'\} \times U_2) \overset{\epsilon}{\sim} z_2$. $\dots$ Inductively, we have a sequence $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n$ (relatively open) and $t_1, \dots, t_n \in \mathcal{T}$ such that $t_j(\{x'\} \times U_n) \overset{\epsilon}{\sim} z_j, \forall j \in \{1, 2, \dots, n\}$. Hence $U_n \subseteq V_\epsilon$. This means that V_ϵ is dense in $\overline{u\pi^{-1}(y)}$.

Let $\Gamma = \bigcap_{n=1}^\infty V_{1/n}$. Then Γ is a residual set of $\overline{u\pi^{-1}(y)}$, and for all $x \in \Gamma$, we have $\overline{\mathcal{T}(x', x)} = R_\pi^n$. In particular, there exists a transitive point $(x_1, x_2, \dots, x_n)$ of R_π^n with $x_1, x_2, \dots, x_n \in \pi^{-1}(y)$. The proof is completed. $\square$

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